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*Simulation of Stream Networks by Headward Growth and Branching**

One of the most interesting accomplishments of recent geomorphic investigations has been the development of a number of techniques and theories which closely simulate or predict a number of the statistical properties of stream networks, including relationships between number, length, and drainage area of streams and the order of the stream network. These methods of stream network generation may be classified according to whether they involve theoretical deductions from a set of axioms (Type 1) or involve induction from the results of a process of simulation (Type 2). An additional dichotomy may be made between methods which are topological only (Type A) and those which are constructed upon a hypothetical drainage area (Type B).

The topological theories of Shreve [*15*, *16*, *17*] and Scheidegger [*10*, *11*, *12*] are of Type 1-A. The mixed-hexagonal theory of Woldenberg [*23*] might be considered to be of Type 1-B. Liao and Scheidegger [*6*] use topological simulation (2-A) to find average values of bifurcation ratios in random tree networks. Finally, the random walk techniques used by Leopold and Langbein [*4*], Schenck [*13*], and Smart, Surkan, and Considine [*20*] are examples of Type 2-B.

All of these methods of simulation either do not presuppose a method of development of the stream network (e.g., Shreve [*15*, *16*, *17*]), or postulate the growth of a stream network by coalescence of streams, and thus involve generation of a stream network starting from the heads of the first-order streams and continuing through the establishment of high-order master streams (e.g., the random walk technique and the branching theory of Scheidegger [*10*]). One exception to this approach was the proposition by Woldenberg [*22*, *23*] that stream networks may originate by allometric growth, although Woldenberg did not develop a process for generation of stream systems by growth.

In the present paper operational methods for the production of stream systems by growth are examined. In a growth and branching process the higher-

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order master streams are the first to be established and the low-order tributaries are formed subsequently.

Headward growth and branching are probably reasonable approximations to the development of certain natural stream systems. The badlands at Perth Amboy, New Jersey, described by Schumm [*14*] developed by lateral encroachment of badland topography upon a flat-topped industrial fill. Carter and Chorley [*2*] found a similar situation during the dissection of a river terrace. Investigations by the author in the Henry Mountains region of Utah suggest that Wisconsin-age pediments have been dissected into intricate badlands by the headward movement of a "wave of dissection." Although a stream network is a unidirectional network in terms of direction of stream flow, the effects of changes of base level must be transmitted upstream, so that stream networks resulting primarily from dissection because of a lowering of base level might be reasonably approximated by a headward growth model.

The models introduced within this paper not only result in networks which satisfactorily simulate the statistical properties of natural stream systems, but in addition they model the formation of the stream network in real time, unlike any of the above-mentioned models.

Two types of growth models are discussed. Each involves both a strictly topological realization and one involving growth within a given drainage area.

In the analysis of stream networks generated by the growth processes, the Strahler system of stream ordering [*21*] is used.

Model I

Simulation techniques grouped under Model I involve growth of a stream network by successive generations. In each generation every active stream may continue unchanged, branch, or become inactive (terminate). The probabilities of branching, termination or continuation unchanged are either direct or indirect parameters of the model. Which of the above events occurs at each stream during a generation is resolved by comparison of random numbers with the probability parameters of the model.

Topological Model

In the topological realization of Model I the stream network begins with a single stream. At the first generation the stream either branches with probability Pr(2), continues unchanged (Pr(1)), or terminates (Pr(0)). If two streams are created by branching, during the next generation each stream independently may branch, terminate, or continue. Streams which terminate during one generation remain inactive during successive generations. This process may be continued indefinitely until the entire network becomes inactive, or until it reaches a predetermined size. This topological model, as well as those to be considered subsequently, are subject to random proportional

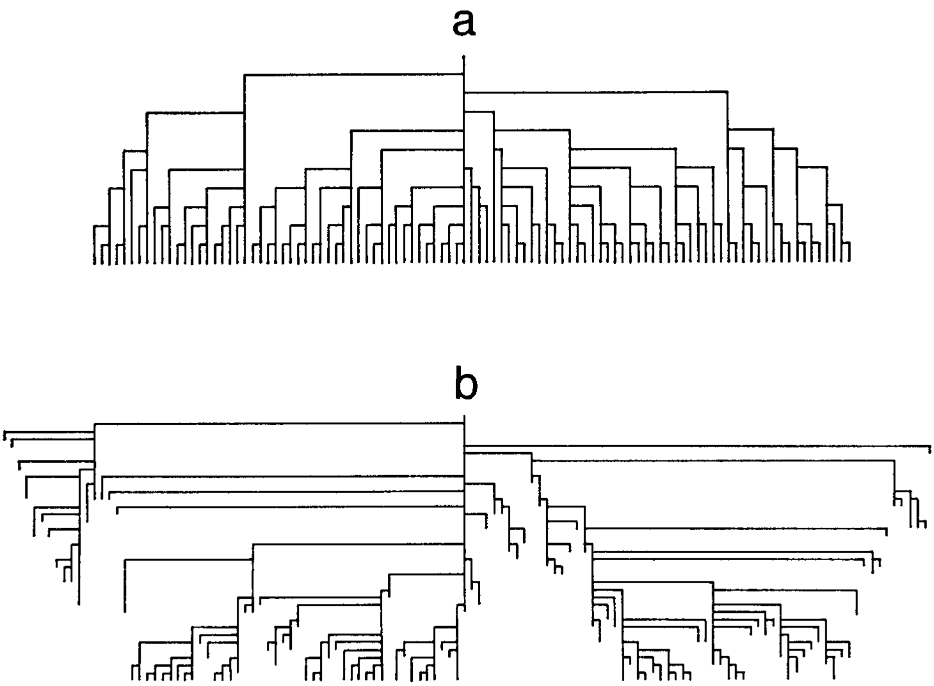


FIG. 1. Topological Patterns by Random Branching during Successive Generations by (a) a Pure Growth Process with 0.5 Probability of Branching and (b) 0.3 Probability of Branching and 0.2 Probability of Termination.

growth (Woldenberg [*23*, p. 104]) because the probability parameters are equal for each active stream and do not change during successive generations.

A special case in this topological model occurs when the probability of termination is zero and the simulation becomes a pure growth process. Figure 1a shows a 5th-order stream network simulated with $\Pr(0) = 0$ and $\Pr(2) = 0.5$.

In the pure growth model the bifurcation ratio of the generated streams is obviously dependent upon the probability of branching ($\Pr(2)$). If $\Pr(2)$ equals unity the bifurcation ratio will equal 2.0 because each stream bifurcates at each generation. If $\Pr(2)$ is less than 1.0 the bifurcation ratio in general will be greater than 2.0 because some streams will continue unchanged for one or more generations and therefore may enter a stream of higher order.

Random headward growth produces networks with numerical relationships significantly different from the random topological model proposed by Shreve [*15*] which assumes that natural stream networks may be topologically random in a special sense: in a random sample of stream networks with a given number of first-order tributaries all possible distinct topological arrangements of these tributaries are hypothesized to be equally probable. Such networks are termed topologically random networks (abbreviated T.R.N.). Particularly simple numerical relationships emerge from networks which are randomly sampled from an *infinite* T.R.N. (I.T.R.N.) [*16*, *17*]. Networks of order Ω sampled from an I.T.R.N. have on the average $2^{\Omega-1} - 1$ tributaries discharging into them from the sides, of which $2^{\Omega-\omega-1}$ will be of order ω [*17*]. Samples of low-order basins from a network of much higher order should be samples from an I.T.R.N. if the network is a T.R.N. Several simulations of fifth- and sixth-order networks were made with Pr(2) of 0.8, 0.5, and 0.2, and the number of tributary streams in second- and third-order basins from the larger networks were determined. The following table gives the average tributary numbers, their standard deviation, and the sample size, compared with expected values for an I.T.R.N.:

	Pr(2)	1.0	0.8	0.5	0.2	ITRN
			Second Order			
$N1$	avg.	2.0	2.18	2.46	2.63	3.0
	s.d.	0.0	.177	.432	.627	?
	no.	—	268	472	412	—
			Third Order			
$N1$	avg.	4.0	5.13	6.65	7.55	11.0
	s.d.	0.0	1.57	2.40	2.76	?
	no.	—	111	170	142	—
$N2$	avg.	2.0	2.35	2.64	2.70	3.0
	s.d.	0.0	.548	.859	.872	?
	no.	—	111	170	142	—

Numerous statistical tests have been performed which confirm that natural stream networks are very nearly topologically random according to Shreve's model [*15, 16, 18, 19*]. Therefore, it is evident from the above table that the pure growth version of Model I results in average stream numbers (and bifurcation ratios) which are rather low compared to those of natural streams. The reason for this is evident from Figure 1a; few first- or second-order streams join directly to streams of 4th or 5th order because each stream is active throughout the entire generating process, whereas a significant number of low-order tributaries join high-order streams in natural drainage basins [*3*]. The probability is negligible for one stream formed at a bifurcation remaining unbranched while the other tributary subsequently becomes of high order.

If natural streams do in some cases develop by a headward growth process, then the explanation for the large numbers of low-order tributaries directly entering high-order streams must be that some streams become wholly or partially inactive after their initial formation. This may be modeled in the topological simulations by introducing a finite probability of death or termination (Pr(0)) among the possible events at each active stream head during each generation of growth; therefore two independent parameters (Pr(0) and Pr(2)) must be specified to determine the pattern of stream growth. Five 5th and 6th order networks were generated with Pr(0) = 0.2 and Pr(2) = 0.3. One of these networks is shown in Figure 1b. Second- and third-order basins in these networks have average tributary numbers higher than the pure growth model but lower than basins sampled from an I.T.R.N.:

	Second Order	Third Order	
	$N1$	$N1$	$N2$
avg.	2.90	9.45	2.76
s.d.	1.18	4.38	1.08
no.	267	84	84

Despite the more realistic bifurcation ratios obtained by inclusion of a probability of termination, the topological approach to simulation of the growth process has inherent weaknesses. Firstly, two probabilities must be specified during the simulation. How these parameters could be related to physical constraints within natural stream systems is uncertain.

A more critical fault of the topological growth model is that streams in nature cannot originate by proportional growth as is assumed in the topological models. Natural streams do not continue to grow indefinitely, but cease growth when all available upland has been dissected, i.e., when stream growth either reaches the primary upland divide or when extension of neighboring streams creates a divide between them. But the effects of divides or competition among stream systems is not felt simultaneously by all stream heads within the network, but is rather felt selectively, depending upon the position of the stream within the network.

Simulation of stream growth upon a given area removes many of the faults of the topological approach and simulates not only bifurcation ratios, but also the relationships between stream length and drainage area to stream order. Drainage basin shape and the effects of boundary conditions may also be investigated using areal models.

Areal Growth Model

The areal growth models utilize an idealized drainage area consisting of a matrix of dimension $M \times N$ in which each of the MN matrix locations (excepting drainage exit points discussed below) has one stream segment originating from it at the completion of growth. Stream segments were constrained to flow either north, south, east, or west from each matrix position.

For simplicity of calculation it was assumed that each stream segment has one unit of drainage area directly contributing to the segment (i.e. a uniform drainage density of unity was assumed). Although the junction of more than two streams at a point is rare in nature, triple junctions were allowed in most of the matrix simulations because of ease of programming. In order to test the effect of allowing of triple junctions, other simulations were made for some models in which junction of more than two streams at a point was forbidden.

In the following discussion an *active* matrix location refers to one from which growth may take place. An active location must either contain a stream segment draining it or be a drainage exit. Extension (growth) of stream segments is not allowed from *inactive* locations on the matrix, i.e., those locations without stream segments draining them or those positions drained by a stream segment for which any of the following conditions are true: matrix locations to the east, west, north, and south are already drained by stream segments or are (1) drainage exits or (2) points outside the matrix. If triple junctions are *not* allowed, a matrix location also becomes inactive whenever two streams become tributary to that location. In no case may a stream grow into a matrix location already drained by a stream segment, into a drainage exit, or to a point outside the matrix.

Before the growth process is initiated the size of the matrix and those points which will serve as drainage exits for the stream networks must be specified. At the beginning of growth these are the only active sites on the matrix because all other matrix locations are undrained by a stream segment.

Because the present model involves generations of growth, every active matrix location must be evaluated for the occurrence of growth during each generation. This was accomplished by constructing a random-order serial listing of the coordinates of all matrix locations which was used as the order of access to the matrix locations during the growth process.

During each generation matrix locations are selected by the use of the random serial listing. If the matrix location is inactive it is not further considered, but if the location is an active site then the possibility of growth of the stream network into adjacent squares is examined. Each of the unoccupied matrix locations surrounding the active site are examined in a random, serial order for the possibility of stream growth from the active location into the unoccupied location. Whether or not the unoccupied matrix location receives a stream growing from the active site is resolved according to the probability of growth specified as a parameter at the beginning of the game by use of random numbers.

The matrix locations to which growth takes place do not become active sites until the subsequent generation. Therefore during each generation only one unit length of growth is possible in a given direction from each active matrix location.

The process of growth is carried through successive generations until all matrix locations excepting the drainage exits have streams draining them. The generated stream networks are then evaluated for stream order and the relevant statistical data for each order of stream are collected. These data for the present model (designated A-I) and for other models discussed below are summarized in Tables 1–3.

Only three conditions, the probability of growth, the location of drainage exits, and the dimensions of the matrix are adjustable in this model. All stream systems generated with this and subsequent models were conducted on a 40 $\times$ 40 matrix, and all simulations with this model allowed drainage exits at all matrix points on the edge of the matrix. In a subsequent model, discussed below, drainage exit conditions were also varied.

Variation of the probability of growth has a great effect on the type of drainage network generated (Figure 2 and Tables 1–3). As the probability of branching decreases from near one to near zero the form of the generated drainage basins changes from long and narrow to irregularly pear shaped and the range in sizes and order of the drainage basins increases. All of these basins, however, show a pronounced symmetry about the center of the matrix, and

TABLE 1

MEAN AND STANDARD DEVIATION OF DRAINAGE BASIN PARAMETERS FOR SECOND-ORDER BASINS*

	1	2		4	5	6	7	8	9	10	11	12
	(66)	(93)	(104)	(93)	(110)	(116)	(97)	(101)	(99)	(348)	(297)	(264)
*N*1	4.77	3.62	3.44	4.20	3.35	3.17	3.20	3.48	3.23	3.12	3.09	3.18
	3.58	*1.91*	*1.93*	*2.76*	*1.66*	*1.52*	*1.88*	*2.00*	*1.66*	*1.49*	*1.40*	*1.51*
*L*1, *A*1	2.07	2.00	1.68	1.62	1.63	1.67	1.80	1.87	1.96	1.65	1.95	2.07
	.947	*.854*	*.707*	*.556*	*.621*	*.627*	*.713*	*.858*	*.755*	*.563*	*.789*	*.772*
*L*2	7.86	4.25	3.81	3.60	3.55	3.23	3.58	3.89	3.63	3.30	4.13	4.41
	4.78	*3.02*	*3.33*	*2.75*	*2.66*	*2.45*	*3.23*	*3.51*	*2.74*	*2.54*	*3.70*	*3.79*
*L*2/*L*1	4.70	2.50	2.37	2.43	2.35	2.12	2.19	2.43	2.03	2.19	2.38	2.44
	3.64	*2.04*	*2.00*	*1.95*	*1.79*	*1.60*	*2.13*	*2.41*	*1.53*	*1.80*	*1.96*	*2.26*
*A*2	16.8	11.2	9.63	10.3	9.00	8.69	9.32	10.3	9.97	8.44	10.1	10.9
	10.0	*6.31*	*7.04*	*7.05*	*5.60*	*5.41*	*6.58*	*7.19*	*6.18*	*5.00*	*5.93*	*7.04*
*A*2/*A*1	9.46	6.13	5.82	6.62	5.70	5.37	5.39	5.92	5.27	5.32	5.49	5.63
	6.66	*3.66*	*3.79*	*4.51*	*3.24*	*3.01*	*3.78*	*4.10*	*3.01*	*3.11*	*3.05*	*3.51*

Column	Model	Drainage Outlets (E = Edge; P = Point)	Probability of Growth	Maximum Junctions at a Point
1	A-I	4-E	0.9	3
2	A-I	4-E	0.5	3
3	A-I	4-E	0.1	3
4	A-II	4-E	1.0	3
5	A-II	4-E	0.5	3
6	A-II	4-E	0.1	3
7	A-II	1-E	0.5	3
8	A-II	1-P	0.1	3
9	A-II-A	4-E	—	3
10	A-II	4-E	0.5	2
11	A-II-A	4-E	—	2
12	RW	4-E	—	2

* Key to table entries: Numbers in parentheses = number in sample; non-italic numbers = average value; numbers in italics = standard deviation.

TABLE 2

MEAN AND STANDARD DEVIATION OF DRAINAGE BASIN PARAMETERS FOR THIRD-ORDER BASINS*

	1	2	3	4	5	6	7	8	9	10	11	12
	(16)	(21)	(25)	(23)	(22)	(26)	(45)	(25)	(22)	(84)	(72)	(59)
*N*1	11.2	15.3	13.5	16.1	15.1	13.7	13.6	18.1	14.3	11.6	11.1	11.7
	4.12	*6.85*	*7.38*	*7.41*	*6.24*	*9.29*	*10.1*	*12.0*	*7.58*	*6.38*	*4.46*	*5.52*
*N*2	2.38	3.00	3.00	2.96	3.36	3.27	3.20	3.92	2.86	3.05	2.76	2.94
	.619	*1.26*	*1.29*	*1.15*	*1.40*	*1.69*	*2.06*	*1.98*	*1.13*	*1.39*	*.889*	*.996*
*N*1/*N*2	4.84	5.19	4.41	5.65	4.61	4.01	4.16	4.51	4.90	3.74	4.03	3.93
	1.87	*1.40*	*1.50*	*2.23*	*1.35*	*1.05*	*1.26*	*1.55*	*1.57*	*.899*	*1.15*	*1.11*
*L*1, *A*1	1.78	1.88	1.55	1.53	1.73	1.62	1.74	1.77	1.91	1.63	1.87	2.04
	.526	*.309*	*.264*	*.176*	*.326*	*.207*	*.302*	*.283*	*.255*	*.274*	*.389*	*.433*
*L*2	3.29	4.92	3.25	3.44	3.75	3.06	3.33	4.10	4.48	3.04	4.22	4.24
	1.44	*2.29*	*1.42*	*1.48*	*1.37*	*1.55*	*1.73*	*2.09*	*2.41*	*1.34*	*1.94*	*2.22*
*L*3	10.4	7.10	6.64	7.22	7.27	6.65	8.64	9.64	7.91	6.55	7.55	8.66
	3.46	*4.98*	*3.84*	*5.99*	*4.41*	*4.94*	*7.88*	*6.81*	*6.23*	*4.43*	*4.72*	*5.76*
	1	2	3	4	5	6	7	8	9	10	11	12
*L*2/*L*1	1.96	2.77	2.09	2.25	2.22	1.92	2.05	2.32	2.39	1.89	2.34	1.79
	.792	*1.80*	*.850*	*.907*	*.821*	*1.04*	*1.27*	*1.08*	*1.33*	*.847*	*1.15*	*1.12*
*L*3/*L*2	4.25	1.75	2.64	2.96	2.44	2.53	3.08	3.01	2.67	2.56	2.33	2.66
	3.03	*1.51*	*2.24*	*3.37*	*2.23*	*1.51*	*2.70*	*2.69*	*3.07*	*1.86*	*2.11*	*1.29*
*A*2	8.61	12.4	8.48	9.74	9.44	8.12	8.82	10.8	11.5	7.94	10.1	10.8
	3.61	*3.69*	*3.23*	*4.42*	*3.11*	*2.73*	*2.97*	*4.60*	*4.89*	*2.63*	*3.79*	*4.22*
*A*3	37.2	49.0	38.4	42.0	44.8	39.9	43.4	59.0	47.4	34.8	40.0	44.7
	8.80	*19.6*	*21.7*	*19.3*	*17.0*	*28.0*	*32.0*	*37.2*	*23.5*	*19.6*	*17.6*	*21.4*
*A*2/*A*1	5.00	6.76	5.43	6.31	5.54	5.06	5.23	6.04	6.10	4.88	5.47	5.40
	1.80	*2.74*	*1.83*	*2.48*	*1.72*	*1.81*	*2.06*	*2.21*	*2.64*	*1.52*	*1.89*	*1.90*
*A*3/*A*2	4.96	4.29	4.60	4.79	5.19	4.76	5.15	5.68	4.58	4.51	4.16	4.35
	2.25	*2.34*	*2.15*	*2.69*	*2.51*	*2.45*	*3.76*	*2.99*	*2.81*	*2.31*	*1.77*	*2.02*

* See key to Table 1.

TABLE 3

CHARACTERISTICS OF BASINS DEVELOPED ON 40 x 40 AREAL MATRIX*

	1	2	3	4	5	6	7	8	9	10	11	12
	Total Number of Streams											
	(5)	(5)	(5)	(5)	(5)	(5)	(2)	(2)	(3)	(3)	(3)	(3)
$N1$	365	482	511	565	517	511	499	513	448	497	425	393
	14.6	*9.40*	*12.6*	*26.2*	*11.0*	*9.15*	*4.95*	*14.1*	*9.45*	*5.69*	*17.6*	*2.89*
$N2$	63.6	95.4	103	96.6	109	105	100	84.0	93.0	116	99.0	88.0
	4.77	*2.70*	*9.15*	*6.11*	*3.19*	*7.56*	*4.24*	*17.0*	*8.72*	*5.57*	*4.36*	*2.00*
$N3$	3.20	19.0	23.8	20.0	24.4	23.8	22.5	19.5	20.7	28.0	24.0	19.7
	.447	*1.22*	*3.70*	*2.45*	*1.82*	*1.64*	*.707*	*3.54*	*.577*	*2.00*	*3.46*	*2.08*
$N4$	—	0.60	3.60	3.20	4.80	4.00	5.00	5.50	3.67	4.00	3.00	3.67
	—	*.894*	*.548*	*.447*	*.837*	*.707*	*1.41*	*.707*	*1.53*	*1.00*	*1.00*	*.577*
$N5$	—	—	—	—	—	0.20	1.50	1.00	0.33	0.67	—	0.33
	—	—	—	—	—	*.447*	*.707*	*0.00*	*.577*	*.577*	—	*.577*
	1	2	3	4	5	6	7	8	9	10	11	12
	First-Order Stream Length											
	(369)	(474)	(509)	(580)	(503)	(500)	(495)	(503)	(451)	(1490)	(1274)	(1178)
$L1$	2.26	1.88	1.67	1.60	1.71	1.65	1.69	1.79	2.02	1.68	1.96	1.99
	1.91	*1.22*	*1.02*	*.918*	*1.08*	*.934*	*1.05*	*1.24*	*1.44*	*1.01*	*1.55*	*1.40*
L_r	.442	.532	.600	.625	.585	.606	.592	.560	.496	.595	.510	.503
										Interior Link Length		
										(1204)	(937)	(1003)
									L_{int}	1.53	1.90	1.98
										.925	*1.54*	*1.39*
									L_r	.654	.526	.505

* For Key see Table 1.

because of the drainage exits ringing the matrix, the drainage network so developed resembles island drainage patterns on homogeneous lithology. Because of the outlet conditions the shape of drainage basins is determined entirely by competition between neighboring stream systems. Statistically significant changes in the average number, length, and drainage area of streams occur as the probability of growth is varied (Tables 1–3).

The behavior of the growth process for high values of the probability of growth is very sensitive to the arrangement of drainage exits and matrix shape. With a probability of growth of unity in the case of a matrix surrounded by drainage exits the final growth product will be four sets of parallel, unbranched streams meeting at their headward end. But if only a few drainage exits are present in the matrix, the stream network will have to expand to fill the matrix so that the drainage pattern will be in general a branched network.

The simulation of growth at near-unity probability under the present model is probably an unsatisfactory representation of the development of natural stream systems in which growth and branching are important. Rates of growth and branching in natural streams should form a continuum of values, whereas in the present model growth in each allowed direction at active matrix locations is either none or one unit. If the probability of growth in the above model

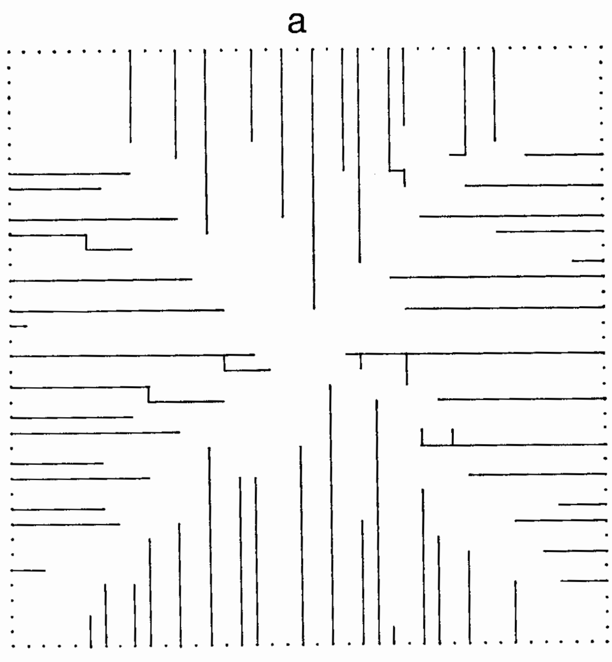


FIG. 2. Stream Networks Simulated by Headward Growth and Branching (Model A-I) on a 40 × 40 Matrix with (a) 0.9 Probability of Growth, (b) 0.5 Probability of Growth, and (c) 0.1 Probability of Growth. Only Streams of Greater than First Order are Shown. Three Junctions at a Point are Allowed. All Matrix Edges have Potential Drainage Exits.

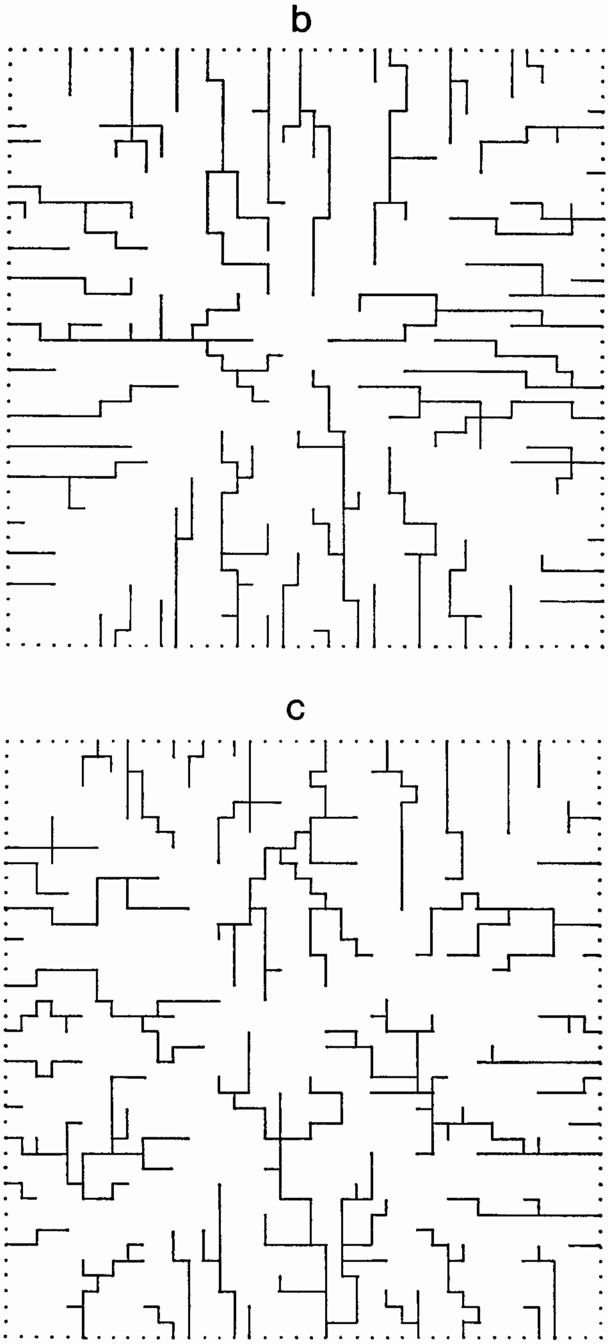


Fig. 2b & c

is reduced to a low value, the average rate of growth and the frequency of branching become more variable and approach a continuous distribution. Simultaneously, the correlation of rates of growth and frequency of branching of adjacent streams is reduced. Furthermore, the direct influence of drainage outlet conditions lessens. The following models further approach the natural continuum by reducing each generation to the events occurring at a single stream head.

Model II

The condition of restricting the events during each generation to one matrix location is simulated by continued random selection of stream heads with determination of growth, inactivity, or continuation at that location.

Topological Models

The procedure in the simulation of stream growth in this model is basically similar to the previous topologic growth model except that branching or death occurs independently in a random sequence on the active stream heads.

A pure growth process [Pr(0) = 0] may be simulated by continued random selection of one stream among all existing streams, each stream having equal probability of selection. This stream is allowed to branch while all other streams remain inactive. This process is continued until a network of desired size is obtained. This model thus has the conceptual simplicity of having no adjustable parameters, for all selected streams branch.

Second- and third-order basins sampled from eleven 4th- and 5th-order networks generated by the above pure growth process resulted in stream numbers and bifurcation ratios less than the expected values for an I.T.R.N.:

	Second Order	Third Order	
	$N1$	$N1$	$N2$
avg.	2.88	8.71	2.90
s.d.	1.08	3.44	.915
no.	175	59	59

Higher bifurcation ratios and a distribution of branches more similar to those of natural streams would result if death (inactivity) were considered as an alternative possibility during the random selection of streams. Here, however, an adjustable parameter, the probability of termination, is introduced into the model, and the topological model is still subject to the criticisms outlined previously.

The Areal Model

The areal model which essentially corresponds to the topological model described above is modified from the previous areal model in only one respect;

after stream growth at one matrix location results in extension of a stream to a new matrix location, that new location becomes immediately active rather than waiting until all other active locations have been sampled. This then has the effect of reducing each generation to the stream growth from a single matrix point. Simulations were conducted for several values of the probability parameter where three junctions are allowed at a point, and for a single case (0.5 probability of growth) where only two junctions were permitted.

Unlike the topological model, the probability of growth remains a parameter in the areal model, and variation of this parameter has an effect upon the statistics of the generated networks (Model A-II in Tables 1–3). However, as is evident from the tables, the effects of variation of the probability parameter are not as marked as with model A-I, and it would have none at all if the possible outcomes at each active matrix site (zero through three directions of growth) were reduced to the dichotomy of the topological model (branching versus no branching). As would be expected the statistical results of model A-II with low probability of growth are virtually identical to those of model A-I with low probability of growth.

Two final modifications may be made to the areal growth model in order to improve the simulation of natural stream growth and at the same time eliminate the variable parameter giving the probability of growth. The first of these is to completely randomize access to the matrix locations so that one matrix location may be sampled several times before an adjacent matrix location is sampled once, thereby reducing correlation between events at adjacent stream locations. Furthermore stream growth at each matrix location during each sampling is limited to only one of the possible directions of growth. Which of these directions is chosen for growth is selected at random. If no possible directions of growth exist at the selected point, a new matrix point is selected, and a new matrix point is also selected following growth in a single direction at a point. Drainage networks were generated with a maximum of both two and three allowed junctions at a point. The networks produced by this process (A-II-A in Tables 1–3) have drainage basin shapes and arrangements nearly identical to those of models A-I and A-II for low probabilities of growth except for a tendency for a more irregular shape. The average stream lengths for lower-order streams is increased significantly, so that the number and bifurcation ratios of 4th and 5th order streams is reduced compared to models A-I and A-II.

Effects of Variations of Boundary Conditions upon Random Growth Stream Networks

For a few simulations the outlet drainage conditions were changed so that some exterior matrix locations were not drainage exits but rather primary drainage divides. A matrix size of 40 $\times$ 40 was retained.

One simulation used one side of the exterior matrix locations as drainage exits and the other three sides as divides. The resulting drainage basins (Fig. 3a) show marked elongation perpendicular to the edge of the drainage matrix

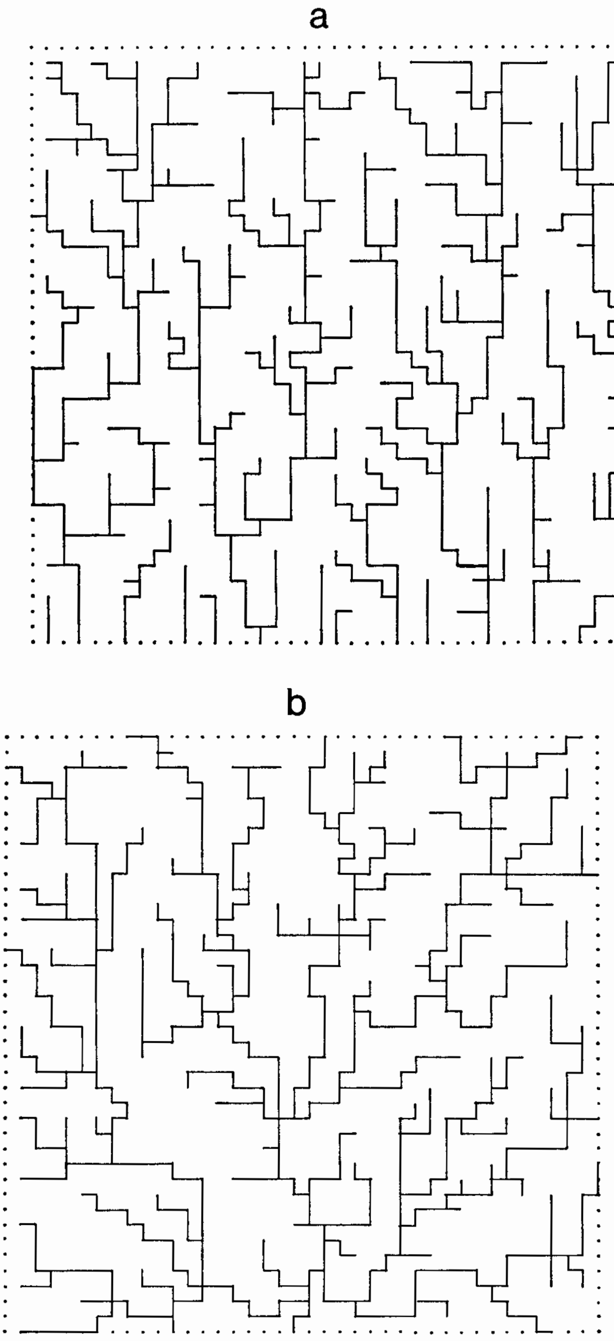


FIG. 3. Stream Networks Generated by Model A-II with (a) Only One Edge of Matrix Having Potential Drainage Exits (0.5 Probability of Growth) and (b) Only One Matrix Location Acting as a Drainage Exit (0.1 Probability of Growth). Three Junctions at a Point Allowed.

having drainage exits, which contrast greatly with those generated with all exterior points as drainage exits. The elongation of the drainage basins results in high values of stream length for streams of high order and minor changes in other statistical parameters.

Another simulation with only a single drainage point in the entire matrix located in the middle of one edge produced a palmate drainage pattern (Fig. 3b) with high length ratios in all order streams (Tables 1–3).

As a result of these simulations under changes of configuration of primary drainage divides and drainage exits, it is apparent that the geometry of the drainage area as a whole may have important effects upon the pattern of drainage that develops and upon its statistical measures.

Effect of Restrictions on Number of Junctions at a Point upon Resulting Networks

For simulations made on a square matrix, junction of a maximum of either two or three streams at a point may be allowed. The effect of this choice on resulting networks for areal growth was investigated for two models (A-II-A and A-II with a 0.5 probability of branching); simulations with each model were conducted with the only change being the maximum number of junctions allowed. The differences in the properties of areal simulations made by the choice of allowed junctions were largely minor (cf. columns 5 and 10 and columns 9 and 11 in Tables 1–3). However, only the models disallowing three junctions may be subject to the topologic tests to follow.

Comparison of Areal Random Growth Models with the Random Walk Model

Five random walk simulations of drainage networks were conducted on a 40 × 40 matrix with all outside edges of the matrix as drainage exits. The simulations were carried out in a manner similar to those made by Schenck [*13*] and Smart, Surkan, and Considine [*20*]. The reader is referred to these papers for details of the methods of generation of random walk stream networks.

In random walk networks the location for a stream source is selected at random and the stream develops in a downstream direction by successive unit extensions in random directions until the stream reaches a drainage exit or an existing stream. Loops in the stream network are not allowed, and triple junctions were forbidden in the present simulations. The probabilities of extensions to the east, west, north, and south were equal. In addition the junction of one stream with the upstream end of an existing stream (source junction) was allowed.

The random walk networks stand in sharp contrast to the growth models because of their asymmetry of development (Fig. 4). The random walk networks are not affected by a drainage exit until the stream actually intersects either a drainage exit or an existing stream. In contrast, the headward growth model

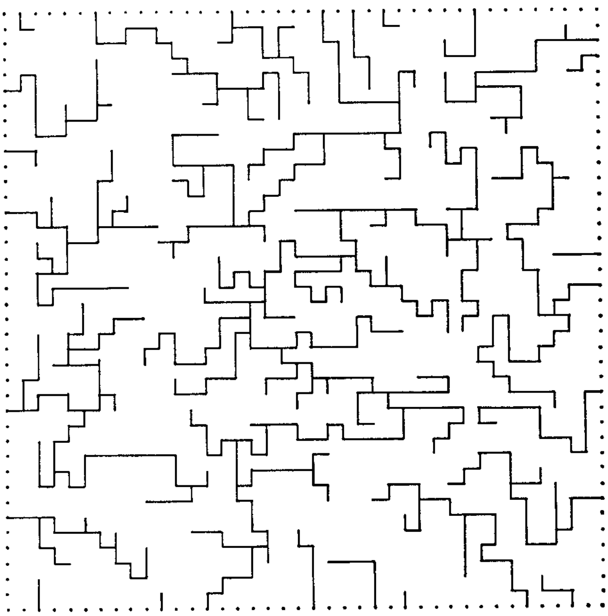


FIG. 4. Stream Network Generated by Random Walk (RW)

begins at the drainage exit and develops headward, and so is more closely affected by the number and arrangement of drainage exits. The drainage basin statistics of the random walk networks (abbreviated RW in Tables 1–3) show little difference from the areal growth models excepting in longer mean stream lengths and fewer total streams on a 40 × 40 matrix.

Topological Randomness of Areal Stream Patterns

The next paragraphs present several statistical tests of the conformance of the areally generated networks to the random topological model of Shreve [*15*, *16*, *17*]. These tests are applied only to networks generated with the restriction that not more than two streams may join at a point.

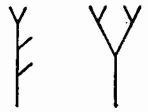
Samples of streams with 6 or fewer first-order tributaries may be feasibly grouped into *ambilateral classes* (topologic classes with left and right handedness neglected) for comparison with expected frequencies for T.R.N. [*19*]. Table 4 shows the ambilateral classification of networks with 4 to 6 first-order tributaries with observed and expected frequencies for the simulation models, the number of T.R.N. in each class, and chi-square tests for goodness-of-fit. Only one rejection of the hypothesis of topological randomness may be made at a 95% level of significance (L.O.S.). Model A-II-A, however, gives consistently better fit then do either the random walk or model A-II (0.5).

Larger stream networks may be tested on the basis of the probability of occurrence of the particular set of numbers of 2nd-order, 3rd-order, etc. streams

TABLE 4

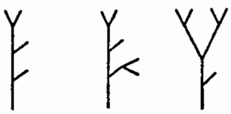
Topological Test for Randomness of Stream Networks with 4 to 6 Tributaries, with Networks Grouped According to Ambilateral Classes*

N1 = 4



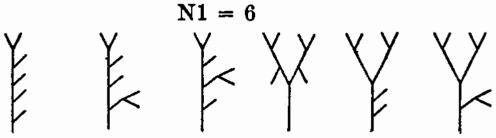
No. of T.R.N.	4	1	
A-II (0.5)	93	10	6.69(3.84)
	82.5	*20.6*	
A-II-A	84	13	2.68(3.84)
	77.6	*19.4*	
RW	84	13	2.68(3.84)
	77.6	*19.4*	

N1 = 5



No. of T.R.N.	8	4	2	
A-II (0.5)	61	23	6	5.69(5.99)
	51.5	*25.7*	*12.9*	
A-II-A	38	19	6	2.67(5.99)
	36.0	*18.0*	*9.00*	
RW	45	10	9	5.73(5.99)
	36.6	*18.3*	*9.20*	

N1 = 6



No. of T.R.N.	16	8	8	4	4	2	
A-II (0.5)	31	10	17	5	5	0	6.86(11.07)
	25.9	*13.0*	*13.0*	*6.5*	*6.5*	*3.2*	
A-II-A	17	12	12	5	5	0	3.82(11.07)
	19.4	*9.7*	*9.7*	*4.9*	*4.9*	*2.4*	
RW	19	13	4	4	7	0	8.33(11.07)
	17.9	*9.0*	*9.0*	*4.5*	*4.5*	*2.2*	

* Sample entries: 93 = observed frequencies; *3.2* = expected frequencies for topologically random network (T.R.N.); 5.73(5.99) = chi-square statistic and critical value.

in a network with a given number of first-order tributaries [*15*, *19*]. All basins exiting to the edge of the matrices with more than 10 first-order tributaries were tested on this basis by a goodness-of-fit test to a quartile rank of the probability of occurrence of the given stream numbers (see discussion of method by Smart [*19*]). The hypothesis of topological randomness could not be rejected for any of the three models tested at a 95% L.O.S.

It is also useful to determine how closely the properties of the generated networks approach those of samples from an I.T.R.N. Shreve [*16*] points out that the frequency of selection of streams of Strahler order from an I.T.R.N. is:

$$f(\omega) = 3/4^{\omega}, \qquad = 1, 2, 3, \cdots \tag{1}$$

Table 5 shows total numbers of streams of order ω generated by the three models compared with expected frequencies for the same number of streams from an I.T.R.N. In all cases a goodness-of-fit test rejects the hypothesis that these streams came from an I.T.R.N. However, in all cases rejection results primarily from a paucity of streams of order greater than three. This is reasonable, because insufficient space is present on the model matrix for generation of many high-order streams. However, the numbers of streams of third and lower orders appear to be close to expected frequencies from an I.T.R.N., and the ratios N1/N2 and N2/N3 are close to the value of 4 expected for an I.T.R.N. The random walk model seems to generate low-order networks least likely to be sampled from an I.T.R.N.

As discussed before, average stream numbers for second- and third-order streams in an I.T.R.N. are:

2nd order: N1 = 3; N2 = 1
3rd order: N1 = 11; N2 = 3; N1 = 1

The hypothesis that the number of 1st-order tributaries in second- and third-order basins is 3 and 11, respectively, cannot be rejected at a 95% L.O.S. for any of the three methods of network generation tested, although the null hypothesis is barely accepted for second-order basins in the random walk

TABLE 5

TEST OF HYPOTHESIS THAT TOTAL NUMBERS OF NETWORKS OF DIFFERENT ORDERS IN SIMULATIONS ARISE FROM A RANDOM SAMPLING OF AN INFINITE TOPOLOGICALLY RANDOM NETWORK

	Order 1	2	3	4	>4	
A-II (0.5)	1490	348	84	12	2	
	1452	*363*	*90.8*	*22.7*	*7.6*	11.24(9.49)
	[.994]	[.619]	[.502]	[5.03]	[4.10]	
A-II-A	1274	297	72	9	0	
	1239	*310*	*77.4*	*19.4*	*6.5*	13.89(9.49)
	[.989]	[.525]	[.382]	[5.54]	[6.50]	
RW	1178	264	59	11	1	
	1135	*284*	*70.9*	*17.7*	*5.9*	11.65(9.49)
	[1.65]	[1.37]	[2.00]	[2.55]	[4.08]	

Key: same as for Table 4, except that figures in brackets are individual chi-square values.

model. Tests for the null hypothesis that the mean number of second-order tributaries in third-order basins is equal to 3 resulted in rejection only in the case of model A-II-A at a 95 % L.O.S., but the margin of rejection was narrow.

In conclusion, both the random growth and random walk models of stream network generation produce stream networks which are very nearly topologically random. For simulations on a 40 $\times$ 40 matrix the drainage area is sufficient that second- and third-order stream networks, but not higher-order basins, may be considered to be sampled from an I.T.R.N.

Stream Lengths and Drainage Areas

Smart, Surkan, and Considine [20] show that lengths of exterior links (first-order streams) and interior links (portions of streams between junctions) in randomly-generated stream networks with discrete stream lengths may be approximated by a geometric probability distribution:

$$f(L) = L_r(1 - L_r)^{(L-1)}, \quad L = 1, 2, 3, \cdots, \tag{2}$$

where L_r is a parameter and $f(L)$ is the frequency of observation of a stream of length L. The maximum-likelihood estimator of L_r is the reciprocal of the average link length (interior or exterior); using this estimate (Table 3), a chi-square test was utilized to determine the goodness-of-fit of equation (2) to the observed distributions of length.

For simulations in which as many as three streams could join at a point the fit of the geometric distribution was tested against only the exterior link lengths, because interior link lengths of zero were permitted. Interior link lengths were tested, however, in the simulations in which no more than two junctions at a point were allowed. In all cases the geometric distribution gave an acceptable fit to the observed frequencies at a 95 % L.O.S.

Except in the case of model A-II (0.5), the average interior link length could not be proven significantly different from the average exterior link length at a 95 % L.O.S. Even in the case of model A-II (0.5) the ratio of average exterior to average interior link length was only 1.1.

If it may be assumed that streams are sampled from an I.T.R.N. and that the lengths of interior and exterior links are independent, random variables from the same population, then the ratios $\bar{L}_{\omega+1}/\bar{L}_\omega$ (where $\bar{L}_\omega$ is the mean length of streams of order ω) approach a value of two [*16*, *18*]. As noted previously, the average length of interior and exterior links was nearly equal in the three simulations made subject to the constraint that no more than two junctions occur at a point. Furthermore, if the drainage density is assumed to be uniform (as is assumed to be exactly true in the simulations), then the area ratios approach the values shown in the following table [*18*]. The values expected under the above assumptions for length and area ratios are compared with the ratios observed for streams of order less than four in the 40 $\times$ 40 matrices:

	$\bar{L}2/\bar{L}1$	$\bar{L}3/\bar{L}2$	$\bar{A}2/\bar{A}1$	$\bar{A}3/\bar{A}2$
Expected	2	2	5	21
A-II (0.5)	1.97	1.98	5.02	20.7
A-II-A	2.11	1.83	5.17	20.4
RW	2.22	1.96	5.48	22.4

The agreement appears to be very good, so that the models tested also appear to satisfy the assumptions made by Smart [*18*] regarding stream length and drainage density.

Summary and Conclusions

Under some conditions the development of a stream network on newly-exposed land may primarily result from headward growth of streams accompanied by intermittent branching to form new streams. For example, such conditions might arise in the dissection of terraces or pediments as a result of stream erosion.

In this paper random growth and branching models have been developed to simulate patterns of growth in natural stream networks from birth to completion. Similar real-time models might be developed to simulate the development or evolution of natural stream systems by other processes than headward growth (e.g., stream capture).

The several topologic growth models discussed produce tributary numbers which are abnormally low for natural stream networks. This is because the assumption of proportional growth made in the topologic models cannot be strictly true in natural streams, for the rate of growth in developing stream systems must vary both through time and at any given time at different locations within the stream network because of competition between adjacent streams for available drainage area.

More satisfactory results in growth simulation are obtained using models in which growth takes place on a given matrix which corresponds to an area of dissected upland. Not only are more realistic tributary numbers obtained, but good predictions of area and length ratios result from the use of areal growth models. The developed networks appear to be topologically random. The simulated networks also clearly show the results of competition between adjacent streams for undissected drainage area.

Several versions of the areal growth models were discussed, with Model A-II-A presumably modelling best the continuum of values of rates of growth and branching in natural stream networks. However, many stream networks may not originate by headward growth, and those that do may eventually be better described by some other model of headward growth than those discussed here. Modifications within existing stream networks by such processes as stream capture are not considered by the growth models.

The geometry of the drainage area subject to stream growth affects the size, shape, and statistics of the resultant stream networks. However, the effects of modifications of boundary conditions upon the number, length, and area ratios is rather minor compared with differences in size, shape, number, and order of the generated basins. The relative insensitivity of bifurcation, length, and area ratios under a wide variety of drainage basin configurations suggests that these dimensionless ratios are not particularly efficient indices of differences between the geometry of natural or simulated drainage basins; these ratios are not strongly correlated with the hydrologic characteristics of stream networks [*1*, *7*], and most authors relating hydrologic variables to drainage basin parameters ignore these dimensionless ratios in favor of dimensional measures of basin size, shape, and gradient [*8*, *9*].

The areal random growth models introduced successfully simulate many of the characteristics of natural stream systems, but it remains uncertain to what extent inferences may be drawn about the processes responsible for natural stream systems on the basis of this intriguing correspondence. One caution about such inferences arises from the wide spectrum of models (including the random walk) which result in numerical relationships among network properties similar to those of natural streams. If, as seems likely, any number of generating processes are able to do as well in simulation, then the argument from simulation process to natural processes may be dangerous (cf. [*5*, p. 421]).

A related question concerns the relevance of randomness in simulation models to processes in natural stream systems. Almost all of the successful simulation models of stream networks involve probabilistic decisions between alternate stream locations or between possible actions at a stream location. Furthermore, natural stream systems appear to exhibit numerical relationships nearly identical to those of topologically random networks [*15*, *16*, *18*, *19*]. Because the processes which act on a geomorphic scale are basically deterministic, the apparent randomness arises from independent variation of a large number of factors affecting the development of stream networks, such as microclimate, lithology, tectonic forces, climatic changes, and historical changes produced by exposure of new strata by erosion, inter alia. In any natural stream system the effect of these variations of process and controls in time and space is to produce variations in any given stream property which may be simulated by a random variable. In fact, it would be impossible to select from nature a stream network in which all of the factors affecting the pattern of the stream network have been isotropic in space and constant through time. However, the most satisfying future model of stream development will be one in which the most factors are accounted for and in which randomness plays the smallest part, although the largely random models will continue as a yardstick against which systematic variations of the properties of natural streams and simulation models may be compared.

Literature Cited

1. Anderson, H. W. "Flood Frequencies and Sedimentation from Forest Watersheds," *Transactions of American Geophysical Union*, 30 (1949), 567–84.
2. Carter, C. S. and R. J. Chorley. "Early Slope Development in an Expanding Stream System," *Geological Magazine*, 98 (1961), 117–30.

3. Giusti, E. V. and W. J. Schneider. "The Distribution of Branches in River Networks," *United States Geological Survey, Professional Paper* 422-G. 1965.
4. Leopold, L. B. and W. B. Langbein. "The Concept of Entropy in Landscape Evolution," *United States Geological Survey, Professional Paper* 500-A. 1962.
5. Leopold, L. B., M. G. Wolman, and J. P. Miller. *Fluvial Processes in Geomorphology*. San Francisco: Freeman, 1964.
6. Liao, K. H. and A. E. Scheidegger. "A Computer Model for Some Branching-Type Phenomena in Hydrology," *Bulletin of International Association of Scientific Hydrology*, 13 (1968), 5–13.
7. Maxwell, J. C. "Quantitative Geomorphology of the San Dimas Experimental Forest, California," *Office of Naval Research, Report No. 19, ONR Project 389-042*. Columbia University, 1960.
8. Melton, M. A. "An Analysis of the Relations among Elements of Climate, Surface Properties, and Geomorphology," *Office of Naval Research, Report No. 11, ONR Project 389-042*. Columbia University, 1957.
9. Morisawa, M. E. "Quantitative Geomorphology of Some Watersheds in the Appalachian Plateau," *Bulletin of Geological Society of America*, 73 (1962), 1025–46.
10. Schneidegger, A. E. "Stochastic Branching Processes and the Law of Stream Orders," *Water Resources Research*, 2 (1966), 199–203.
11. ———. "Random Graph Patterns of Drainage Basins." *Paper 157, 14th Assembly International Union of Geodesy and Geophysics, International Association of Scientific Hydrology*. (Berne, 1967).
12. ———. "Horton's Law of Stream Numbers," *Water Resources Research*, 4 (1968), 655–58.
13. Schenck, H. S., Jr. "Simulation of the Evolution of Drainage Basin Networks with a Digital Computer," *Journal of Geophysical Research*, 68 (1963), 5739–45.
14. Schumm, S. A. "Evolution of Drainage System and Slopes in Badlands at Perth Amboy, New Jersey," *Bulletin Geological Society of America*, 67 (1956), 597–646.
15. Shreve, R. L. "Statistical Law of Stream Numbers," *Journal of Geology*, 74 (1966), 17–37.
16. ———. "Infinite Topologically Random Channel Networks," *Journal of Geology*, 75 (1967), 178–86.
17. ———. "Stream Lengths and Basin Areas in Topologically Random Channel Networks," *Journal of Geology*, 77 (1969), 397–414.
18. Smart, J. S. "Statistical Properties of Stream Lengths," *Water Resources Research*, 5 (1968), 1001–14.
19. ———. "Topological Properties of Channel Networks," *Bulletin of Geological Society of America*, 80 (1969), 1757–73.
20. Smart, J. S., A. J. Surkan, and J. P. Considine. "Digital Simulation of Channel Networks," *Proceedings of International Association of Scientific Hydrology, 14th International Union of Geodesy and Geophysics*. (Berne, 1967), 87–98.
21. Strahler, A. N. "Quantitative Analysis of Watershed Geomorphology," *Transactions of American Geophysical Union*, 38 (1957), 913–20.
22. Woldenberg, M. J. "Horton's Laws Justified in Terms of Allometric Growth and Steady State in Open Systems," *Bulletin of Geological Society of America*, 77 (1966), 431–34.
23. ———. "Spatial Order in Fluvial Systems: Horton's Laws Derived from Mixed Hexagonal Hierarchies of Drainage Basin Areas," *Bulletin of Geological Society of America*, 80 (1969), 97–112.