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Theoretical Model of Optimal Drainage Networks

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A simulation model of drainage network optimization is presented in which channels shift to minimize total stream power ρgQS within the network. The simulation model starts from an arbitrary initial stream network developed on a square matrix, such as produced by random headward growth. Discrete stream capture then is simulated within the network, occurring wherever a new stream linkage would produce a steeper course than the original. Such capture produces a network with minimum power optimization but flow directions constrained to eight directions. Individual segment end points are then allowed to migrate by iterative relaxation with a direction and rapidity of motion governed by the gradient of stream power at the node. This valley migration is subject to the constraint that the sources and outlet remain fixed. The resulting networks are visually and morphometrically more similar to natural stream networks than the original networks produced by the random headward growth model.

1. Introduction

Theoretical treatment of drainage network morphology has been based upon the seemingly contradictory approaches of stochastic and optimal modeling. *Horton*'s [1932, 1945] development of stream ordering and a model for stream junction angles was based upon a deterministic and optimal concept of drainage network growth. *Shreve*'s [1966, 1967] random topology model revolutionized study of drainage network morphology and shifted attention to a stochastic paradigm that has dominated research for almost 25 years. Nonetheless, important aspects of network morphology exhibit consistent, nonrandom patterns in areas of uniform lithology and minor structural controls. Chief among these are the tendency for uniformity of drainage density and regularities of shape and orientation properties, particularly junction angles [*Horton*, 1932; *Lubowe*, 1964; *Howard*, 1971*c*; *Pieri*, 1980; *Roy*, 1983, 1984]. The present paper presents a model for drainage network evolution that emphasizes spatial optimization of a stochastically generated network. The paper suggests that important optimization processes control development of drainage basin shape and that study of such processes has been neglected. A companion paper [*Howard*, 1990] shows that these systematic variations in basin form have important effects upon hydrologic response.

Drainage basin networks exhibit four basic qualities: topology (linkage properties), scale (drainage density; length and area relationships), orientation (direction and shape properties of streams, slopes, and divides), and relief (gradient and hypsometric relationships). Most quantitative studies of drainage networks have concerned only one or two of these qualities, although strong interrelationships occur between all aspects of drainage network organization [*Howard*, 1971*b*; *Abrahams*, 1984*a*]. Most attention has been given to the topological and scale properties of drainage networks. Shreve's random model and its scale extensions [*Shreve*, 1966, 1967, 1969; *Smart*, 1968, 1978; *Smart and Werner*, 1976] have been very successful in explaining drainage network patterns. Of the four qualities, orientation properties have been least explored.

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Studies of orientation have concerned structural control on channel direction [*Judson and Andrews*, 1955; *Jarvis*, 1976; *Bannister*, 1980], divide angles [*Flint*, 1980; *Abrahams*, 1980; *Abrahams and Flint*, 1983] and stream junction angles [*Horton*, 1932; *Lubowe*, 1964; Howard, 1971*c*; *Pieri*, 1980; *Roy*, 1983, 1984]. Structural controls involve local constraints on absolute stream directions, whereas the studies of divide and junction angles concern universal constraints on relative orientations. Studies of basin shape have primarily concentrated on development of morphometric indices for measurement of shape variations [*Horton*, 1932; *Miller*, 1953; *Schumm*, 1956; *Chorley et al.*, 1957; *Strahler*, 1964].

Stream junction angles are strongly related to other drainage basin properties. As pointed out by *Horton* [1932] and *Howard* [1971*c*], junction angles are influenced by channel gradient relationships at the junction. *Pieri* [1980] uses junction angles to infer gradient relationships in Martian channel networks. Channel orientations at junctions also affect the topology of stream junctions in at least two ways: first, by restricting the relative number of low-order tributaries entering the interior (acute) sides of two confluent streams upstream from the junction [*Flint*, 1980; *Abrahams and Flint*, 1983] and second, through the effects of direction changes at junctions enhancing the frequency of relatively large tributaries on the outside of bends [*Abrahams*, 1984*b*].

This paper introduces a simulation model of the temporal evolution of orientation relationships in a drainage network which suggests that streams erode their valley walls toward an optimal pattern, a process that is here termed valley migration. *Band* [1987] also has suggested that gradual shifting of stream channels occurs in drainage basins; however, his model involves forcing by relative amounts of sediment delivery from sideslopes rather than by the mechanism of differential stream erosion presented here.

Optimal network models arise in numerous contexts. Examples include transportation and communication systems (for example, railroads, telephone networks, and water distribution systems), biological systems (for example, the human circulation system), the structural geometry of organisms (for example, the hexagonal structure of honeycombs), and solid mechanics (for example, cooling cracks). Many theoretical models of optimal networks have simple analytical solutions for individual nodes (such as the minimum power junction angle criterion discussed below), but global (network-wide) analytical solutions are generally possible only for the simplest cases, for example, hexagonal honeycombs and cooling cracks. As with the present model, iterative solutions are generally necessary, with the method of solution depending upon the particular type of model.

2. Junction Angle Adjustment in Stream Networks

Over the past few years evidence has accumulated that streams adjust their angles of junction towards optimum values determined by the relative gradients, widths, discharges, and other characteristics of the merging streams. However, the degree and manner in which such adjustments affect the overall spatial structure of the drainage network has remained unclear. The modeling effort reported below suggests that systematic lateral channel erosion (including, but not limited to, adjustments at junctions) has a profound effect upon orientation relationships within stream networks and that this valley migration, together with stream capture, may be the primary mechanisms through which orientation and relief characteristics affect topological and scale-dependent properties.

Four important processes affect the planimetric structure of drainage basins: (1) development of new channel links, often by headward growth; (2) change of basin size by migration of divides; (3) discrete capture of one channel by another; and (4) valley migration and junction angle adjustments. Studies of developing drainage networks generally show rapid creation of a rudimentary drainage network that is strongly influenced by original topography and structural controls. After the initial drainage network has formed, migration of divides, capture, and valley migration slowly operate to form a more regular network, with little change in overall drainage density if climate, exposed lithology and downcutting rate remain constant. The random headward growth model [*Howard*, 1971*a*] is used here as an approximation to initial development of a drainage network with strong local influences. These networks are then operated on by a stream capture model [*Howard*, 1971*b*], whose effects include both divide migration and discrete capture. Valley migration is the final process, and may also result indirectly in deletion or creation of interior or exterior Shreve links. The sequential operation of the three models is clearly an idealization, since all of these processes act to some degree concurrently.

Although the model developed here is not solely a criterion for stream junction angles, it is based upon previous junction angle models, which are reviewed briefly below.

2.1. *Extant Models of Junction Angles*

Junction angle models have been based upon three types of criteria: geometric relationships at junctions [*Horton*, 1932; *Howard*, 1971*c*], momentum-balance [*Mosley*, 1976], and optimality principles [*Howard*, 1971*c*; *Roy*, 1983, 1984].

2.1.1. *Geometric models.* The earliest quantitative model of stream junction angles proposed by *Horton* [1932] suggests that the junction angle of a tributary with a main stream is determined at the time the tributary forms through erosion by runoff flowing along the path of steepest gradient on the adjacent hillslope. This produces a junction angle, ψ, which is determined by the ratio of the gradient of the main stream, S_m, to that of the tributary, S_t (Figure 1*a*):

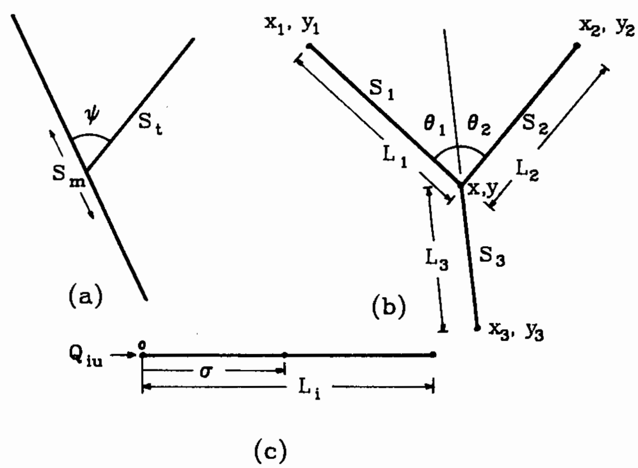

Fig. 1. Junction angle and stream power definitions: (*a*) Horton model; (*b*) geometric and minimum power models; (*c*) definition of power integration along a channel segment, with upstream to left.

$$\cos \psi = S_m/S_t \tag{1}$$

Data from *Lubowe* [1964] showed that this criterion is a reasonable predictor of junction angles if gradients are averaged by stream order.

Howard [1971*c*] noted that this equation predicts that two streams of equal gradient should never join, and he suggested that the criterion be revised such that two junction angles θ_1 and θ_2 be defined that individually satisfy an equation of the above form using the ratios of the gradients of the downstream continuation of the merged streams S_3 to the gradients of the individual entering streams S_1 and S_2 (Figure 1*b*):

$$\cos \theta_1 = S_3/S_1 \qquad \cos \theta_2 = S_3/S_2 \tag{2}$$

Howard [1971*c*] showed that these formulas are more successful that (1) in predicting junction angles in natural networks with tributaries of nearly equal size. However, *Abrahams* [1984*a*] found that (1) may be less biased than (2) when the merging streams are of significantly different sizes.

Howard [1971*c*] also suggested that gradients adjust dynamically during the evolution of the channel network in response to fluvial processes acting at the junction. He postulated that channels not meeting at the angles given by (2) would be subject to bank erosion and sedimentation that would tend to force compliance with (2). However, the criterion expressed in (2) was based upon a crude model of junction geometry and was only sketchily related to the assumed channel processes.

2.1.2. *Momentum balance model.* *Mosley* [1976] suggested that junction angles tend to conserve lateral momentum of the incoming flows, that is,

$$Q_1V_1 \sin \theta_1 = Q_2V_2 \sin \theta_2 \tag{3}$$

where V_1 and V_2 are average flow velocities in the tributaries. This provides a criterion only for the relative magnitudes of θ_1 and θ_2. A similar relationship can be written for conservation of downstream momentum, but streams having velocities that are constant or increasing downstream cannot conserve downstream momentum.

2.1.3. *Optimal models.* Another approach is based upon determining the junction angles that minimize some objective function, or "cost" associated with the junction. The optimal solution is formulated by assuming that the stream passes through three points (x_i, y_i; Figure 1*b*), and the problem is to find the junction location (x, y) that gives the minimum total cost Ω given by the sum over the three channel segments of the costs per unit channel length C_i times the segment lengths L_i:

$$\Omega = \sum_{i=1}^{3} C_i L_i \tag{4}$$

Roy [1983] showed that this implies junction angles that are independent of the lengths L_i:

$$\begin{aligned} \cos\theta_1 &= (C_3^2 + C_1^2 - C_2^2)/2C_1C_3 \\ \cos\theta_2 &= (C_3^2 + C_2^2 - C_1^2)/2C_2C_3 \end{aligned} \tag{5}$$

Howard [1971*c*] had previously found this solution for the particular case of a minimum power (or minimum rate of work) cost function.

$$C_i = \rho g Q_i S_i \tag{6}$$

where Q_i is the discharge flowing through the channel segment with gradient S_i, ρ is the fluid density, and g the gravitational constant. Similar minimum power solutions for junction angles of arteries were discovered by *Murray* [1926]. An extensive literature on optimal junctions in biology is summarized by *Woldenberg and Horsfield* [1983, 1986].

This model for stream junctions was generalized by *Roy* [1983, 1984] to include optimality criteria other than stream power. *Roy* [1983] identified minimum total flow resistance ($\rho gWRS/V_2$, where W is channel width and R is the hydraulic radius), minimum resisting force $\rho gWRS$, and minimum channel volume WR as alternative cost functions. Other possibilities include minimum shear stress ρgRS, minimum stream power per unit width $\rho gQS/W$, and any of the above weighted either directly or inversely by hydraulic radius.

A minimum power criterion is used in this paper. The use of stream power as a criterion for effectiveness of geomorphic processes is not new; *Bagnold* [1957, 1966] relates sediment transport rate to stream power, *Bull* [1979] suggests that stream erosional and transport processes are governed by critical threshold of stream power, and *Hickin and Nanson* [1984] and *Nanson and Hickin* [1986] quantify bank erosion rates as a function of stream power.

2.1.4. *Intercomparison of models.* In order to compare junction angles predicted by the three types of models, two equations of downstream hydraulic geometry and three hydraulic relationships are assumed which allow junction angles of all models to be expressed solely in terms of the ratio of discharges of the merging streams and an exponent z relating channel gradient to discharge:

$$W = K_1 Q^{1/2} \tag{7}$$

$$S = K_2 Q^z \tag{8}$$

$$V = K_3 R^{2/3} S^{1/2}/n \tag{9}$$

$$Q = WRV \tag{10}$$

$$Q_3 = Q_1 + Q_2 \tag{11}$$

where n is Manning's resistance parameter and K_1, K_2, and K_3 are assumed to be areally and temporally constant. Note that the channel depth and hydraulic radius are assumed to be essentially equal.

Table 1 compares junction angles predicted by the various junction angle models for a range of assumed values of z and the ratio of discharges of the merging streams, μ:

$$\mu = Q_l/Q_s \tag{12}$$

where the larger discharge is Q_l and the smaller Q_s.

The geometric (2) and minimum power (5) models predict smaller junction angles for z closer to zero and smaller angles for the tributary with greater discharge, with the contrast in angles increasing with greater μ. For values of z in the range of natural stream networks ($-0.2 \geq z \geq -0.6$, see *Howard* [1971*b*] and *Flint* [1974]) the geometric and minimum power models predict angles that are similar enough that measured junction angles from natural streams are unable to distinguish between the models, because of the high variability of natural junction angles. Furthermore, both models predict junction angles that are on the average close to observed junction angles [*Howard*, 1971*c*, *Roy*, 1984].

The several cost functions suggested by *Roy* [1983, 1985] as alternatives to minimum power differ considerably in terms of the dependence of junction angles and cost gradient upon relative stream discharges and profile concavity, z. They generally predict junction angles much different from the minimum power and geometric models and thereby also far from natural junction angles. However, the minimum total flow resistance and minimum resisting force cost criteria predict junction angles reasonably close to the minimum power model for streams with hydraulic geometry given by (7)–(11) and might be alternatives to a minimum power model.

The Horton model (1) predicts only one junction angle, which is equated here with the total junction angle:

$$\psi = \theta_1 + \theta_2 \tag{13}$$

Since the gradients of the two merging streams and their downstream continuation may all be different, it is uncertain how the Horton model should be applied in practice. For the purposes of this comparison the main stream gradient S_m in (1) is equated with the gradient of the continuing stream S_3, and the tributary gradient is the steeper of the two merging streams. The Horton model thus predicts total angles much smaller than the other models for nearly equal discharges of the merging streams, but the geometric, minimum power, and Horton models essentially converge in their predictions for large μ.

The momentum balance model can only be used to predict the ratio of junction angles of the two tributaries (Table 1). For nearly equal discharges the momentum balance model is equivalent in the ratio of angles to the geometric and minimum power models. However, for large discharge ratios, the momentum balance criterion indicates that the stream of larger discharge (the main stream) will be only

TABLE 1. Comparative Predictions of Junction Angles Using Geometric, Minimum Power, and Momentum Criteria

	θ_l		θ_s/θ_l			$\psi = \theta_1 + \theta_2$		
μ	G	MP	G	MP	MOM	G	MP	H
				$z = -0.1$				
1	21.09	21.09	1.00	1.00	1.00	42.18	42.18	21.09
2	16.21	14.62	1.63	1.92	3.33	42.57	42.71	26.37
5	10.91	8.08	3.05	4.55	11.50	44.19	44.82	33.28
10	7.90	4.86	4.82	8.70	26.98	46.01	47.13	38.11
20	5.66	2.82	7.51	16.63	61.61	48.13	49.72	42.48
50	3.60	1.33	13.20	39.02	180.36	51.16	53.22	47.55
100	2.56	0.74	19.89	74.39	404.31	53.48	55.79	50.93
				$z = -0.25$				
1	32.77	32.77	1.00	1.00	1.00	65.53	65.53	32.77
2	25.36	23.63	1.60	1.79	2.98	65.92	66.00	40.55
5	17.17	13.99	2.93	3.85	9.82	67.46	67.89	50.29
10	12.46	8.95	4.55	6.81	22.37	69.15	69.92	56.69
20	8.93	5.57	6.96	11.96	49.52	71.08	72.17	62.15
50	5.10	2.90	13.34	24.90	138.93	73.72	75.11	68.02
100	4.04	1.75	17.73	43.10	301.42	75.65	77.18	71.61
				$z = -0.5$				
1	45.00	45.00	1.00	1.00	1.00	90.00	90.00	45.00
2	35.26	35.26	1.55	1.55	2.49	90.00	90.00	54.74
5	24.10	24.10	2.73	2.73	7.61	90.00	90.00	65.91
10	17.55	17.55	4.13	4.13	16.67	90.00	90.00	72.45
20	12.60	12.60	6.14	6.14	35.51	90.00	90.00	77.40
50	8.05	8.05	10.18	10.18	94.52	90.00	90.00	81.95
100	5.71	5.71	14.76	14.76	196.76	90.00	90.00	84.29
				$z = -1.0$				
1	60.00	60.00	1.00	1.00	1.00	120.00	120.00	60.00
2	48.19	60.00	1.46	1.00	1.82	118.72	120.00	70.53
5	33.56	60.00	2.40	1.00	4.73	113.96	120.00	80.41
10	24.62	60.00	3.44	1.00	9.85	109.40	120.00	84.78
20	17.75	60.00	4.92	1.00	19.89	105.02	120.00	87.27
50	11.37	60.00	7.82	1.00	48.45	100.24	120.00	88.87
100	8.07	60.00	11.08	1.00	93.17	97.50	120.00	89.43

Explanation of symbols: μ, ratio of discharges of larger to smaller tributary; θ_l, junction angle for larger tributary; θ_s/θ_l, ratio of junction angles of smaller to larger tributary; ψ total junction angle; *G* junction angles using geometric criterion (2); *MP* junction angles using minimum power criterion (5); MOM ratio of junction angles using momentum balance criterion (3); *H* total junction angle using Horton criterion (1); and z exponent relating gradient to discharge (8).

slightly bent at the junction, giving junctions similar to the Horton model.

2.2. *Minimum Power Model of Network Modification*

Changes in confluence angles at an individual junction cannot be independent of concurrent adjustments at adjacent junctions. Even when an appropriate criterion for the optimal junction angles for individual junctions is identified, it is not clear a priori how the mutual interadjustments of adjacent junctions would occur. In particular, it is difficult to predict the degree to which space-filling restrictions limit the approach of individual junction angles towards optimality.

In this section the above questions are addressed through a simulation approach which is based upon a constrained optimization of the stream network, including, but not limited to, its angles of junction. In order to utilize the simulation approach a candidate criterion for network geometry optimization must specify for any arbitrary geometry of incoming and outgoing streams at a node both the direction that the node should migrate to reach the optimum value and the relative rapidity of adjustment. Strictly geometric criteria, such as the Horton and Howard models reviewed above, are not suitable since they do not provide measures of direction and rapidity of adjustment. The minimum power model (as well as similar models discussed by *Roy* [1983]) does provide the type of vector measure required for a simulation approach and is used below.

Equation (5) results from expressing L_i in terms of the coordinates, taking the derivatives of (4) with respect to x and y, setting these to zero, and making appropriate trigonometric substitutions. A different approach is taken here. The derivatives of Ω with respect to x and y give a gradient of the power function (the optimal solution being a zero gradient). The direction of the negative of the gradient for any arbitrary location (x, y) points toward the optimal location of the junction (since power is to be minimized), and the assumption is made that the rapidity of movement of the junction point toward the optimum is proportional to the magnitude of the gradient. Note that the total power Ω can be defined analogously for any two straight, intersecting, but unbranched stream segments; however, the minimum power solution will always occur when the two segments form a straight line.

A stream network can be idealized as a set of interconnecting straight segments, some of which intersect at junctions. It is assumed here that junctions involve a maximum of three segments (the two merging streams and their con-

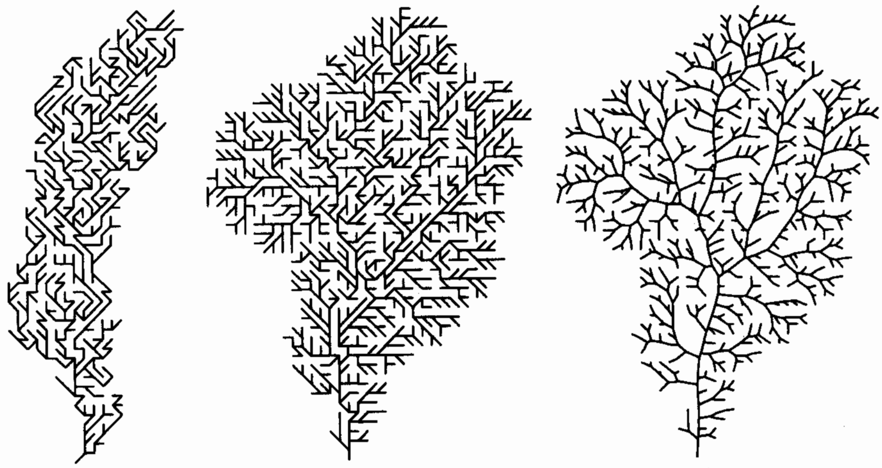

Fig. 2. Evolution of a representative channel network in capture and minimum power models for $z = -0.5$: (*a*) stream network produced by headward growth (part of a drainage pattern developed on a 100×100 matrix); (*b*) headward growth network after modification by systematic capture (note that the drainage area has increased and the basin boundaries have shifted significantly); (*c*) network resulting from minimum power optimization of systematic capture basin.

tinuation). A gradient of power can be individually defined for each junction and each intersection of two segments by fixing the far end points of each segment and using the above approach. The simulation model uses these gradients to change any arbitrary initial stream pattern by stages as the channel pattern evolves towards an optimum (minimum power). Detailed assumptions are outlined below.

2.2.1. *The initial stream network.* The stream networks used in the simulations are portions of networks generated on a 100×100 matrix using a headward growth process such that all networks grow inward from the matrix edges [*Howard*, 1971*a*]. The headward growth model produces networks with topological, scale, and shape properties similar to natural networks and is a better representation of natural drainage networks than downstream random walk models [*Howard*, 1971*a*, *b*]. Figure 2*a* shows a representative headward growth network extracted from the 100×100 matrix. These networks are then subjected to successive captures [*Howard*, 1971*b*] with an assumed value of the exponent z which is equal to the value subsequently used in the minimum power optimization. In the capture model any given segment may be rearranged from its present course to a different course if the downstream gradient along the alternate path is greater than the existing gradient, subject to limitations on the number of streams (two) allowed to join at a given node. Stream profiles are episodically updated using (8) during the capture process. Figure 2*b* shows the stream network resulting from modification of the network shown in Figure 2*a* for a value of z of -0.5. Note that the size and shape of individual drainage basins within the matrix is altered by the capture process.

The growth and capture models are improved versions of the *Howard* [1971*a*, *b*] models in that streams are allowed to flow in eight rather than four directions. Each matrix node is drained by a stream segment so that each node is either the source of a stream, a point of continuation of the stream at which the stream may or may not bend, a stream junction (only two streams are allowed to join at a node), or, at the matrix edges, a network terminating node. Note that in the present paper, one or more segments may be used to represent a "Shreve link", which is defined as either an unbranched tributary a length of stream between junctions [*Shreve*, 1966]. The networks resulting from headward growth followed by stream capture (Figure 2*b*) have topology, length, area, and shape characteristics similar to natural ones [*Howard*, 1971*b*], and they have uniform drainage densities. The major failing of the matrix-simulated networks is the limited number of segment directions, which results in poor simulation of junction angles and other orientation properties.

Howard [1972] showed that the capture model always decreases the total power expended within the channel network for $0.0 > z > -1.0$ and that the capture model results in junction angles that are, on the average, closer to the minimum power junction angle criterion than the angles within the original headward growth network. Thus the capture model can be viewed as an approximation to the optimization model introduced below. This intermediate step of modification of the drainage pattern by capture is used because the optimization model as presently formulated is not well equipped to handle topological changes within the drainage network nor does it assure uniform drainage density.

2.2.2. *The optimization model.* Each stream head (source) is assumed to receive an equal discharge Q_0 at its head. Furthermore, each internal and external segment is assumed to receive lateral drainage at a rate proportional to its length (constant of proportionality α), set to unity in the simulations. Thus the discharge at a point along a segment is equal to the weighted sum of the lengths of all the segments upstream from the point plus Q_0 times the number of sources. A value for Q_0 of 0.3α is assumed here.

2.2.3. *Power calculations.* The power expended along a segment of length L_i is found by integrating the product $\rho g Q S$ along the segment:

$$\Omega_i = \rho g \int_0^{L_i} (Q_i + \alpha\sigma) S(\sigma)\, d\sigma \tag{14}$$

where Q_i is the discharge entering the segment from upstream, σ is the position along the segment measured from its upstream end (Figure 1c), and $S(\sigma)$ is the gradient of the stream at position σ. The simulations assume the validity of (8) and the invariance of α. Thus the power expended along a segment is

$$\Omega_i = \rho g K_2 \int_0^{L_i} (Q_i + \alpha\sigma)^{1+z}\, d\sigma$$

$$= \rho g K_2[(Q_i + \alpha L_i)^{2+z} - Q_i^{2+z}]/[\alpha(2+z)] \quad (15)$$

The total power at a junction or at a node in the middle of a stream is the sum of the power expended along segments entering and leaving the node. The gradient of power in the x direction is calculated by differentiating the total power with respect to x. The gradient in the y direction is found analogously. This can be done explicitly for arbitrary coordinates of the streams entering and leaving the nodes. For example, the x direction gradient for a junction at (x, y) is:

$$\frac{\partial\Omega}{\partial x} = \rho g K_2 \sum_{i=1}^{3} \{(Q_i + \alpha L_i)^{1+z}(x - x_i)/L_i\}$$

$$+ \rho g K_2 \sum_{i=1}^{2} \{[(Q_3 + \alpha L_3)^{1+z} - Q_3^{1+z}](x - x_i)/L_i\} \quad (16)$$

which makes use of the relationships:

$$Q_3 = Q_1 + Q_2 + \alpha L_1 + \alpha L_2 \quad (17)$$

$$L_i = [(x - x_i)^2 + (y - y_i)^2]^{1/2} \quad (18)$$

For use in the simulations (17) and (18) are substituted into (16) to eliminate L_i and Q_3.

2.2.4. *Simulation procedure.* The simulation proceeds by iterations. Prior to each iteration, the lengths and contributing discharges for individual segments are calculated. The x and y gradients of power are calculated for each node using the above approach. Individual nodes are moved in the direction opposite to the gradient of power by an amount equal to the magnitude of the gradient times the factor:

$$K_f L_a / \nabla\Omega_m \quad (19)$$

where $\nabla\Omega_m$ is the maximum gradient of power occurring within the network, L_a is the average segment length, and K_f is a relaxation parameter set to a small enough value that results are independent of its value (0.05 here).

Only interior nodes were allowed to move during the simulation (that is, the sources and the outlet of the network were fixed). This constraint is necessary because otherwise the solution for minimum power is for all segments to have zero length. This constraint roughly corresponds to the tendency in natural drainage basins toward a uniform drainage density.

The simulation continues by successive recalculation and movement of nodes until one of the following conditions occurs:

1. The length of any individual segment reaches a critical lower limit, here set to 0.05 times the average segment length. If it is an exterior segment, it is eliminated. If it is an interior segment it is eliminated if the topology remains unchanged (that is, if at least one end of the segment is not at a junction). If both ends of the segment terminate at a junction, the simulation is concluded.

2. The lateral movement of a segment causes it to override an adjacent node. This occurs only for source nodes, which are fixed. When this occurs the source node and its exterior segment are eliminated, and the simulation continues.

3. The maximum gradient in power within the network falls below a specified fraction of its initial value, say 0.05. However, all of the present simulations terminated as a result of criterion 1.

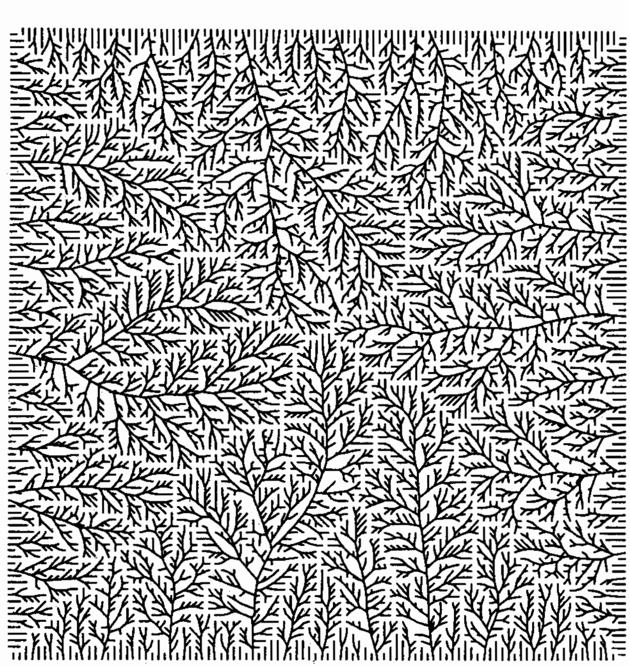

Fig. 3. Drainage pattern developed by the minimum power model on a 100 × 100 matrix for $z = -0.25$. Note that all basins drain toward the four matrix edges.

The case of a segment between junctions approaching zero length presumably corresponds to a situation in which the order of merging of streams would be altered to a new topology, as discussed below. However, by the time such a situation arises the simulations generally have progressed far enough to have produced a strongly optimized network.

2.2.5. *Simulation results.* Simulations were run with z of -0.15, -0.25, -0.5, and -1.0 starting from capture-generated networks created with the equivalent value of z. It is clear that the minimum power simulations modify the original matrix networks in a manner that greatly improves their visual similarity to natural networks in terms of orientation relationships (Figures 2c, 3, 4). The resultant networks exhibit the same types of patterns as noted by *Howard* [1971c] in natural networks in that large-magnitude channels tend to deflect toward the point of junction of tributaries (most noticeably for the larger tributaries), producing a large-scale pseudomeandering. The major difference in the networks generated by different values of z are smaller junction angles and more parallel alignment of channels as the exponent approaches zero.

The simulations point out that individual junction angles

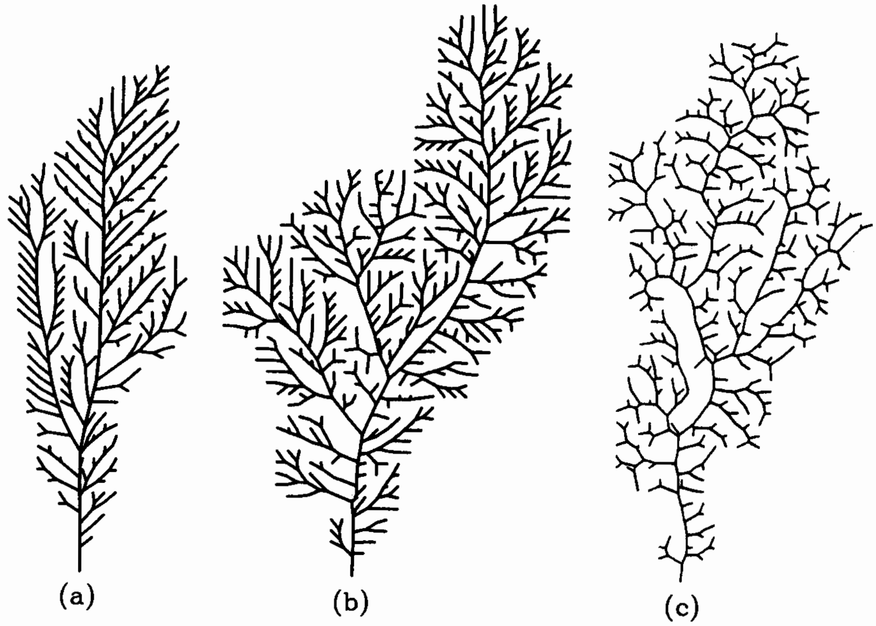

Fig. 4. Comparison of stream networks generated from initial network shown in Figure 2*a* by systematic capture and minimum power optimization with (*a*) $z = -0.1$, (*b*) $z = -0.25$, and (*c*) $z = -0.75$.

can only optimize within the constraints imposed by the location and concomitant adjustment of nearby junctions. It is interesting to enquire whether these constraints create only a variance about the optimum junction angles or whether they force systematic deviations from optimality. Due to the restriction of junction angles in the initial networks to 45° increments, the starting values of θ_l, θ_s, and ψ are highly variable and often appreciably biased relative to the minimum power angles given by (5). In the following discussion the term bias refers to the difference between the angles observed in the simulations and angles predicted by (5). Figure 5 shows the average bias of θ_l, θ_s, ψ, and θ_s/θ_l as a function of the discharge ratio of the incoming streams μ for the initial headward growth network, the network after capture, and finally the network after minimum power optimization. This figure shows results for networks simulated with $z = -0.5$, and similar results are obtained for other values of z.

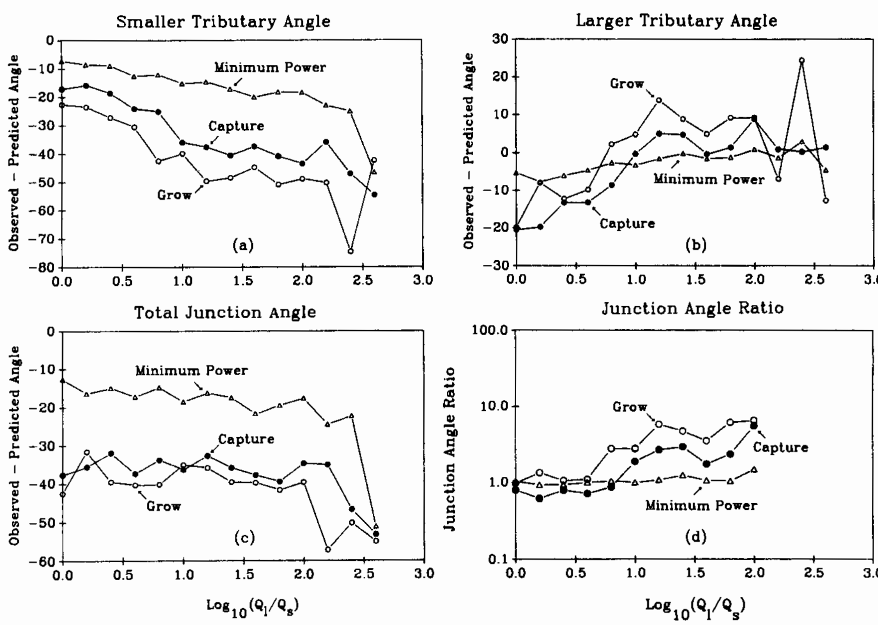

Fig. 5. Characteristics of junction angles generated by headward growth (grow), systematic capture (capture), and minimum power (minimum power) generated with $z = -0.5$, plotted versus the logarithm of the ratio of the discharges of the larger and smaller tributaries μ entering the junction (the larger tributary is defined as the one with the largest discharge). The junction angle produced by the model is the observed angle, whereas the angle given by (5) is the predicted angle. Each data point is the average of values for several junctions, typically decreasing from about 250 for $\log_{10}(\mu) = 0.0$ to 5 for $\log_{10}(\mu) = 2.5$. The junction angle ratio is the observed divided by the predicted values of θ_s/θ_l.

In general, total junction angles become less biased relative to the initial headward growth network as a result of capture modification of the network and still less biased after minimum power optimization. The minimum power optimization produces very rapid convergence of the θ_s/θ_l values toward the model optimum values, even when total junction angles are still strongly biased. This convergence is most pronounced for large values of the discharge ratio. This implies that in junctions of streams of unequal discharge the small junction angles of the large tributary will show less variance from the optimum than the large junction angles of the smaller tributary. In other words, the larger tributary adjusts more rapidly than the smaller tributary.

However, these patterns of adjustment, and the overall appearance of the simulated networks, are strong functions of the exponent z. For values of z only slightly less than zero, such as the network shown in Figure 4*a* created with $z = -0.1$, larger streams are weighted much more strongly than smaller in the power gradient function $\nabla\Omega$ with the result that mainstreams rapidly become nearly straight and adjustment of small tributaries lags well behind. However, as z approaches -1.0 the weighting becomes nearly equal in large and small streams, and mainstreams become more indirect (Figure 4*c*). In fact, for $z = -1.0$ the minimum power solution becomes the classic minimum total pathlength solution with all values of ψ being 120°. Note also that the total junction angle ψ becomes larger as z decreases, as predicted by (5) and shown in Table 1.

2.2.6. *Comparison with natural networks.* The visual similarity of the networks simulated by the minimum power optimization to natural stream networks suggests that a similar optimization occurs in the latter. The minimum power networks are more similar in appearance to natural networks than are purely randomly generated networks, such as random walk [*Leopold and Langbein*, 1962; *Schenck*, 1963; *Smart et al.*, 1967], random capture [*Howard*, 1971*b*], and headward growth [*Howard*, 1971*a*].

However, the slowness of the natural processes of drainage pattern modification and the considerable superimposed irregularity caused by structural influences, stream meandering, inter alia, make direct confirmation of the minimum power model impossible at present. However, a number of indirect lines of evidence suggest that optimization of planform does occur in natural drainage basins. Data cited by *Lubowe* [1964], *Howard* [1971*c*], *Abrahams* [1984], and *Roy* [1984] indicate that natural drainage networks exhibit consistent variation of junction angles as a function of discharge ratio and channel gradients. These variations are generally similar to those predicted by the geometric and minimum power models of junction angles, but no single model explains all of the observed patterns. Nonetheless, the data strongly suggest continual processes of pattern adjustment occur in natural drainage networks.

Howard [1971*b*] collected data on several dimensionless morphometric parameters in drainage basins that suggest that some of these planimetric parameters are functions of basin hypsometry as measured by the exponent z. Figure 6 shows data for five of these parameters for the third-order natural basins measured by Howard as well as the headward growth basins and the minimum power simulated basins. Since the headward growth networks do not assume a particular value of z, the average value of the parameters are shown as a horizontal line in Figure 6. Each of the data points for the natural streams represents an average value of several basins. The data are not strictly comparable because the data for the simulated basins were collected for basins with magnitudes greater than 100. For most of the parameters, results from the minimum power simulations overlie or at least overlap those from natural networks. Also, the simulated minimum power values are generally closer to the natural networks than are those of the initial headward growth simulations. The major exception is the strong bias in exterior-interior link length ratios. This is probably the result of a limitation in the simulation model discussed below.

Overall, therefore simulated networks produced by the sequence of systematic capture and minimum power optimization replicate most of the dimensionless topological and shape parameters measured on natural drainage networks, as well as producing reasonable values of junction angles. This suggests, albeit tenuously, that analogous processes occur in the natural and simulated basins. Suggestions for more direct validation of the model are discussed below.

3. DISCUSSION

The very strong resemblance between natural and the simulated networks suggests that natural networks undergo a similar evolution, modifying their planform by lateral shifting of channels to produce regular junction angles and, between junctions, straight valleys. Natural networks presumably are never as irregular as the initial conditions for the simulations, and much of the lateral shifting probably occurs early in the history of the network when divides are low. Climatic changes, exposure of new rock types, and changes in base level can also cause systematic changes in relative discharges, gradients, and other channel properties which may in turn affect junction angles, producing gradual adjustments of the drainage pattern by lateral shifting of valleys.

If valley migration does occur, it would cause other changes in the drainage basin. In particular, low-order tributaries draining into the side of the main channel toward which the valley was shifting would be shortened and possibly steepened or eliminated, whereas tributaries on the opposing side would be lengthened and possibly elaborated by development of tributaries. Divides would also shift in response to valley shifting. Another occasional effect of the valley migration in the simulations is shorting of channel segments to the point that three streams would join at a point. A more realistic simulation might allow streams entering from opposite sides to move past each other, reversing their order of joining the main stream. If valley shifting in natural networks occurs rapidly enough compared to the rate of downcutting, then the valley shifting could cause differences in opposite side valley wall steepness and the corresponding drainage density of small tributaries.

The remainder of this discussion will be concerned with three related topics: processes in natural stream systems tending to produce optimal geometry and valley migration, possibilities for further model validation, and potential extensions and modifications of the model.

3.1. *Natural Processes of Valley Migration*

Howard [1971*c*] suggested that stream junction angles change as a result of unequal erosion of opposite stream banks or valley walls. He proposed that merging streams have an optimum angular arrangement at which bank erosion

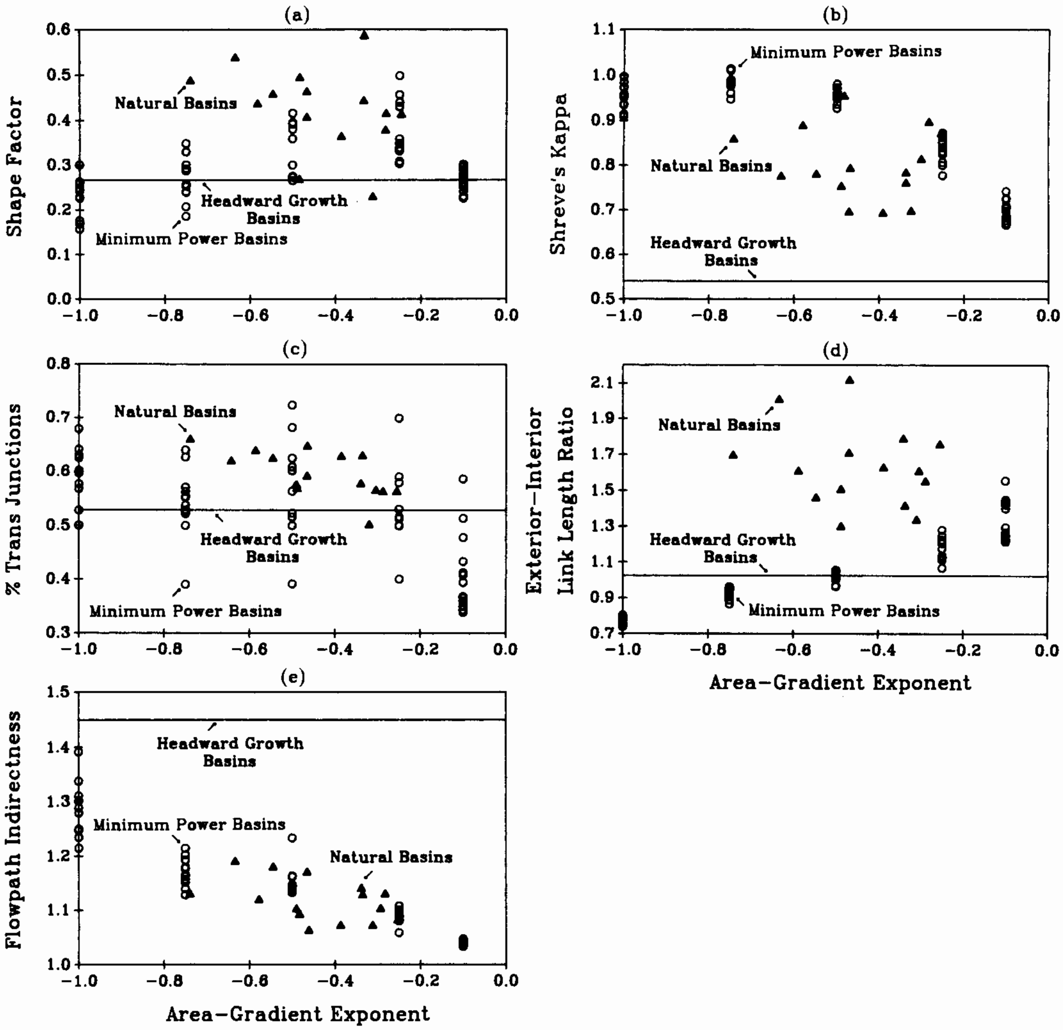

Fig. 6. Comparison of average dimensionless parameters measured in natural stream networks (natural basins) with networks simulated by headward growth (headward growth basins) and by stream capture followed by minimum power optimization (minimum power basins): (*a*) the shape factor is defined is the ratio of the total network area to the square of the distance from the network mouth to the farthest point of the divide [after *Horton*, 1932, p. 351]; (*b*) the Shreve's kappa is the total number of Shreve links (interior and exterior) in the network times the drainage area, divided by the square of the total length of channel [*Shreve*, 1967, p. 185]; (*c*) the percentage of transjunctions is the frequency of occurrence, when following the path of greatest magnitude upstream, of the entrance of successive tributaries from opposite sides of the main channel [*James and Krumbein*, 1969, p. 547]; (*d*) the exterior-interior link length ratio is the ratio of the average length of external links to average interior link length; (*e*) the flowpath indirectness is the pathlength following the stream channel from a given location in the network to its outlet divided by the straight-line distance (In the natural third-order streams this is measured only from the farthest point on the divide from the outlet [*Howard*, 1971*b*], whereas for the simulated streams it is the average of the ratio measured from all interior and exterior nodes).

is balanced and showed that this implies concordant merging of water surface profiles and/or floodplains at the junction. The same processes are suggested here to occur in the more general context of valley migration. If a stream valley is slightly curving, flood flows on the inside bank have a shorter and steeper path. If centrifugal forces are small, the inside bank will be more strongly eroded, tending to straighten the channel. The relative importance of centrifugal forces can be evaluated by taking the ratio of the centrifugal forces to tractive forces in a channel:

$$\frac{V^2\mathscr{C}}{gS} = \frac{R\mathscr{C}}{f} \tag{20}$$

where $\mathscr{C}$ is the channel curvature (inverse of the radius of curvature) and f is a friction factor defined as the ratio of the squares of shear to average velocity. This force ratio is likely to be high in gently sloping fluvial channels with low width-depth ratios and meandering often occurs. However, this ratio is low for steep, wide, shallow channels and for flood flows across floodplains. In such cases, valley migration will be relatively uncomplicated by meandering and junction angles may be fairly regular.

Even where meandering is well developed in the stream channel, the valley on a larger scale may exhibit regular migration and consistent angles of valley junction. When the main stream occupies only a small portion of its valley but migrates back and forth across the valley, the erosion on opposite walls of a curved valley will be unequal over the long run, tending to straighten the valley.

Band [1987] suggested a different mechanism producing valley migration, whereby differences in volume of sediment delivered from opposing valley walls would shift the channels away from the slope delivering the greatest volume. Unfortunately, Band has investigated only a two-dimensional cross-sectional model which cannot be compared with the present model.

3.2. *Model Validation*

Valley migration in most natural stream networks must occur extremely slowly, limiting opportunities for direct validation of the model. The most rapid changes of network pattern generally occur during early stages of stream incision, when divides are low and network patterns are commonly far from optimal. Small networks on easily eroded materials such as rills on slopes, natural or man-induced badlands [*Schumm*, 1956; *Howard and Kerby*, 1983], and exposed lake bottoms [*Morisawa*, 1964] offer opportunities for detailed observations of network evolution. Controlled experiments such as those conducted by Schumm and his colleagues [*Schumm et al.*, 1987] may be even more diagnostic. Such rapidly evolving networks have been examined for many years, and temporal changes in network pattern have been documented. However, the minimum power and other models of channel and junction angle adjustment all hypothesize that the rate and direction of such changes will be governed in part by gradient relationships within the network. The author is not aware of any studies of drainage pattern evolution that include sufficiently detailed observations of channel gradients to be useful in validation of the present model.

Various indirect lines of evidence may be useful in evaluating the network optimization model. The observations on drainage basin morphometry introduced above (e.g., Figure 6) are one example. More such measurements in natural drainage basins would be useful. In particular, the minimum power models of junction angles and valley migration, as well as most other such optimality criteria, predict that the form of drainage basins should be a strong function of the rate of downstream decrease in channel gradients (e.g., Figure 4 and Table 1).

Other aspects of drainage basin morphology may be indicative of long-term valley migration. *Howard* [1971*c*] suggested that the systematic curvature of stream valleys near junctions may result from gradual changes in basin concavity. If this is the case, the minimum power criterion (or some similar model) can be used to predict the expected direction of migration of individual junctions, which can be compared with observed curvatures. Other possible indices of valley migration may be differences in steepness of valley walls and preferential position of streams in their valleys.

There are several difficulties in using the types of observations mentioned above. Channel gradients as measured from maps are generally biased. Natural valley networks are sufficiently irregular that large sample sizes are required for model testing. Processes contributing to this irregularity can be somewhat systematic, such as channel meandering [*Abrahams*, 1984*b*], or more nearly random, such as structural control or effects of local landslides.

3.3. *Model Improvements and Extensions*

If the network optimization model operates starting from an arbitrary initial network, such as a random headward growth network, lateral shifting of the streams eventually causes any of several occurrences: (1) fixed, first-order nodes may become overrun; (2) exterior segments may become vanishingly shortened; (3) some interior segments not terminating in junctions at both ends may become very short; (4) interior segments bordered by junctions at each end may reach near-zero length; and (5) certain portions of the drainage area may be drained by very few or no segments as channels migrate away or segments are eliminated. As discussed previously, the first three cases are handled in the model simply by eliminating the overrun or shortened segment. However, treatment of the last two cases in the model would require extensive programming and large amounts of computer time. Situation 4 results in topological changes within the network, which requires extensive bookkeeping. Poorly drained areas (case 5) should have new first-order streams initiated into them, which requires both identification of the underdrained areas and decisions about the initial course and length of new tributaries. The use of systematic capture [*Howard*, 1971*b*] as an intermediate step in the simulations allows topological changes leading toward an approximation of minimum power optimization while maintaining uniform drainage density. Ideally, the optimization model would accommodate network changes of type 4 and 5 above so that the capture program would not be needed.

In addition, the gradient toward the optimum solution ($-\nabla\Omega$ here) is generally steep far from the optimum but very gentle near the optimum at $\nabla\Omega = 0$. As a result one might expect that few natural stream networks would be as strongly suboptimal as, say, the random headward growth networks. Strongly suboptimal networks are likely to be young, such as rill networks on fresh roadcuts or exposed lake beds. On the other hand, the small gradient close to the optimum suggests that further change may be very slow and often overshadowed by other systematic, chaotic, or random processes, such as river meandering and structural controls.

The present model assumes that stream gradient is a power function of stream discharge (8). *Howard* [1981] shows that such power functions would occur in alluvial stream networks with areally uniform sediment supply and constant grain size and in bedrock streams eroding at a constant rate in bedrock of uniform erodibility. The occurrence of stream capture and valley migration disturbs this relationship by changing the length or topology of the channel network. The model therefore assumes that the capture and valley migration processes occur very slowly compared with stream downcutting or redistribution of alluvium within the channel network. Furthermore, the power law relationship will be a poor approximation to gradient relationships if downcutting rates, sediment yield, sediment size, or bedrock erodibility vary spatially or temporally within the stream network. However, the capture and/or valley migration models could be incorporated in more general simulation models that account for stream gradient evolution in space and time and incorporate the above effects.

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